Representing Gaussian Processes via Brownian Motion

with Applications to Stochastic Analysis

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Motto: Gaussian processes are difficult, Brownian motion is easy. We show that “almost all” Gaussian processes admit a Fredholm representation with respect to a Brownian motion.

Moral: Analysis is easier, if you model directly via the Fredholm representation, and you lose “almost no” generality.

We extend the Fredholm representation to a transfer principle and develop stochastic analysis by using it. In particular, we prove an Itô formula that is, as far as we know, the most general Skorohod-type Itô formula for Gaussian processes so far.

Finally, we give applications to equivalence in law and series expansions of Gaussian processes.
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2. Fredholm Representation

3. Malliavin Calculus and Skorohod Integrals

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Gaussian Processes
The Very Basics

- $X = (X_t)_{t \in [0, T]}$ is Gaussian if all its finite-dimensional projections $(X_{t_1}, \ldots, X_{t_n})$ are multivariate Gaussian.
- There is a correspondence between symmetric positive semidefinite functions $R$ and centered Gaussian processes.

\[ \Rightarrow \text{Knowing } X \text{ is knowing } R \text{ and knowing } R \text{ is knowing } X. \]

- Some classes of forms of $R(t, s)$:
  1. $f(t \wedge s)$ martingales
  2. $\rho(|t - s|)$ stationary processes
  3. $\nu(t) + \nu(s) - \nu(|t - s|)$ stationary-increment processes
  4. $R(s, u) = R(s, t)R(t, u)/R(t, t)$, $s < t < u$, Markov processes
  5. $R(at, as) = a^{2H}R(t, s)$, $a > 0$, $H$-selfsimilar processes

- The Brownian motion has $R(t, s) = t \wedge s$.

\[ \Rightarrow (1), (3), (4), (5) \text{ and } (6) \text{ (with } H = \frac{1}{2} \text{) above hold.} \]

\[ \Rightarrow \]
1 Gaussian Processes

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FREDHOLM REPRESENTATION
The Theorem

Theorem (Fredholm Representation)

Let $X = (X_t)_{t \in [0, T]}$ be a separable centered Gaussian process. Then there exists a kernel $K_T \in L^2([0, T]^2)$ and a Brownian motion $W = (W_t)_{t \geq 0}$, such that

$$X_t = \int_0^T K_T(t, s) \, dW_s$$

if and only if the covariance $R$ of $X$ satisfies the trace condition

$$\int_0^T R(t, t) \, dt < \infty.$$
Fredholm Representation
Some General Remarks

- The Fredholm Kernel $K_T$ usually depends on $T$ even if $R$ does not.
- $K_T$ may be assumed to be symmetric.
- $K_T$ is unique in the sense that if there is another representation with kernel $\tilde{K}_T$, then $\tilde{K}_T = UK_T$ for some unitary operator $U$ on $L^2([0, T])$. (And some other Brownian motion $\tilde{W}$.)
- The Fredholm Representation Theorem holds also for the parameter space $\mathbb{R}_+$, but the trace condition seldom holds, i.e. typically

$$\int_0^\infty R(t, t) \, dt = \infty.$$  

- If the covariance $R$ is degenerate, one needs to extend the probability space to carry the Brownian motion.
Fredholm Representation
Some Square-Root Remarks

- $K_T$ (operator) can be constructed from $R_T$ (operator) as the unique positive symmetric square-root, i.e. the operator $K_T$ is a limit of polynomials:

$$K_T = \lim_{n \to \infty} P_n(R_T).$$

- The positive symmetric square-root is different from the Cholesky square-root. Indeed, the Cholesky square-root would correspond the Volterra Representation theorem

$$X_t = \int_0^t K_T(t, s) \, ds.$$ 

The Volterra representation does not hold for Gaussian processes in general.
Consider a truncated series expansion

\[ X^n_t = \sum_{k=1}^{n} \int_0^t e_k^T(s) \, ds \cdot \xi_k, \]

where \((\xi_k)\) is i.i.d. \(\sim N(0, 1)\) sequence, and \((e_k^T)\) is an orthonormal basis in \(L^2([0, T])\).

\(X^n\) is not purely non-deterministic \((F_{0+}^{X^n} \text{ is not trivial})\) \(\implies X\) cannot have Volterra representation.

On the other hand, by choosing \((e_k^T)\) to be the trigonometric basis on \(L^2([0, T])\), \(X^n\) is a finite-rank approximation of the Karhunen–Loève representation of standard Brownian motion on \([0, T]\).

\(\implies \text{ as } n \to \infty, X^n \to W\) (a Volterra process, obviously).
Fredholm Representation
Example II (Brownian Bridge)

Let $B$ be the Brownian bridge, i.e., formally $B_t = W_t | \{ W_T = 0 \}$. The orthogonal representation of $B$ is

$$B_t = W_t - \frac{t}{T} W_T.$$

$\implies$ $B$ has a Fredholm representation with kernel

$$K_T(t, s) = 1_{[0, t)}(s) - \frac{t}{T}.$$

The canonical representation of the Brownian bridge is

$$B_t = (T - t) \int_0^t \frac{1}{T - s} \, dW_s.$$

$\implies$ $B$ has also a Volterra representation with kernel

$$K_T(t, s) = \frac{T - t}{T - s}.$$
Fredholm Representation
The Proof

The Mercer’s theorem (a.k.a. the eigenvalue decomposition) \( \implies \)

\[
R(t,s) = \sum_{i=1}^{\infty} \lambda_i T e_i^T(t)e_i^T(s),
\]

where \((\lambda_i T)\) and \((e_i^T)\) are the eigenvalues and the eigenfunctions of the covariance operator

\[
R_T f(t) = \int_0^T f(s)R(t,s)\,ds,
\]

and \((e_i^T)\) is an orthonormal basis on \(L^2([0, T])\).

\(R_T\) is a covariance operator \(\implies\) \(R_T\) admits a square-root operator \(K_T\). The trace condition \(\implies\) \(R_T\) is trace-class \(\implies\) \(K_T\) is Hilbert–Schmidt.

\(\implies\) \(K_T\) admits a Kernel.
Indeed,

\[ K_T(t, s) = \sum_{i=1}^{\infty} \sqrt{\lambda_i} e_i^T(t)e_i^T(s). \]

Now \( K_T \) is obviously symmetric and we have (by simple calculations)

\[ R(t, s) = \int_0^T K_T(t, u)K_T(s, u) \, du. \]

\[ \Rightarrow \] the Fredholm Representation follows (in law), and \( \omega \)-by-\( \omega \) by enlarging the probability space, if needed (details omitted, just trust me).
The **Wiener integral** of a function $f$ w.r.t. $X$, denoted by $\int_0^T f(t) \, dX_t$, extends linearly the relation

$$X_t - X_s = \int_0^T 1_{[s,t)}(u) \, dW_u,$$

closed under a suitable norm.

The **Malliavin derivative** of the random variable $\int_0^T f(t) \, dX_t$ is the process

$$D_s \int_0^T f(t) \, dX_t = f(s), \quad s \in [0, T].$$

Extend this under a suitable norm and obey the chain rule.

The **Skorohod integral** is the adjoint of the Malliavin derivative. It extends both the Wiener and the Itô integrals.
The adjoint operator $\Gamma^*$ of a kernel $\Gamma \in L^2([0, T]^2)$ is defined by linearly extending the relation

$$\Gamma^* 1_{[0,t]} = \Gamma(t, \cdot).$$

This is not the same adjoint as in the Malliavin–Skorohod case before.

**Remark**

If $\Gamma(\cdot, s)$ is of bounded variation for all $s$ and $f$ is nice enough, then

$$\Gamma^* f(s) = \int_0^T f(t) \Gamma(dt, s).$$
Theorem (Transfer Principle)

Let $X$ be a Gaussian Fredholm process with kernel $K_T$. Let $D_T$, $\delta_T$, $D^W_T$ and $\delta^W_T$ be the Malliavin derivative and the Skorohod integral with respect to $X$ and to the Brownian motion $W$. Then

$$\delta_T = \delta^W_T K^*_T$$

and

$$K^*_T D_T = D^W_T.$$ 

Proof: Trivial.
Theorem (Itô Formula)

Let $X$ be centered Gaussian process with covariance $R$ and let $f \in C^2$. Then

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) \delta_t X_s + \frac{1}{2} \int_0^t f''(X_s) \, dR(s, s),$$

if anything.

Proof: Either trivial, straightforward or extremely technical, depending on the generality required.
Recall the Hitsuda Representation Theorem: A centered Gaussian process $\tilde{W}$ is equivalent to a Brownian motion $W$ if and only if there exists a Volterra kernel $\ell \in L^2([0, T]^2)$ such that

$$d\tilde{W}_t = dW_t + \int_0^t \ell(t, s) dW_s \cdot dt.$$ 

Now, let $\tilde{X}$ and $X$ be Gaussian Fredholm processes with

$$\tilde{X}_t = \int_0^T \tilde{K}_T(t, s) dW_s,$$

$$X_t = \int_0^T K_T(t, s) dW_s.$$
Suppose then that $\tilde{X}$ has (also) representation

$$\tilde{X}_t = \int_0^T K_T(t, s) \, d\tilde{W}_s$$

where $\tilde{W}$ and $W$ are equivalent.

Then, obviously $\tilde{X}$ and $X$ are equivalent. By plugging in the Hitsuda connection we obtain

$$\tilde{X}_t = \int_0^T \left[ K_T(t, s) + \int_s^T K_T(t, u) \ell(u, s) \, du \right] \, dW_s.$$

Thus, we have shown the following:
Theorem (Equivalence of Laws)

Let $X$ and $\tilde{X}$ be two Gaussian process with Fredholm kernels $K_T$ and $\tilde{K}_T$, respectively. If there exists a Volterra kernel $\ell \in L^2([0, T]^2)$ such that

$$\tilde{K}_T(t, s) = K_T(t, s) + \int_s^T K_T(t, u)\ell(u, s) \, du,$$

then $X$ and $\tilde{X}$ are equivalent in law.

If the kernel $K_T$ satisfies an appropriate non-degeneracy property, then the condition above is also necessary.
In the same way, as in the case of equivalence of laws, we see that:

**Theorem (Series representation)**

Let $X$ be a Gaussian Fredholm process with kernel $K_T$ and let $(\varphi^T_k)$ be any orthonormal basis in $L^2([0, T])$. Then

$$X_t = \sum_{k=1}^{\infty} \int_{0}^{T} K_T(t, s) \varphi^T_k(s) \, ds \cdot \xi_k,$$

where $(\xi_k)$ is i.i.d. sequence of standard Gaussian random variables.

The series above converges in $L^2(\Omega)$; and also almost surely uniformly if and only if $X$ is continuous.
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Thank you for listening!

Any questions?