Representing Gaussian Processes via Brownian Motion

with Applications to Stochastic Analysis

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Abstract
and the Moral of the Story

Motto: Gaussian processes are difficult, Brownian motion is easy.

We show that "almost all" Gaussian processes admit a Fredholm representation with respect to a Brownian motion.

Moral: Analysis is easier, if you model directly via the Fredholm representation, and you lose "almost no" generality.

We extend the Fredholm representation to a transfer principle and develop stochastic analysis by using it. In particular, we prove an Itô formula that is, as far as we know, the most general Skorohod-type Itô formula for Gaussian processes so far.

Finally, we give applications to equivalence in law and series expansions of Gaussian processes.

Outline

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- $X = (X_t)_{t \in [0,T]}$ is Gaussian if all its finite-dimensional projections $(X_{t_1}, \ldots, X_{t_n})$ are multivariate Gaussian.
- There is a correspondence between symmetric positive semidefinite functions R and centered Gaussian processes.

$\Longrightarrow$ Knowing X is knowing R and knowing R is knowing X.

- Some classes of forms of $R(t,s)$:
  1. $f(t \wedge s)$ martingales
  2. $\rho(|t-s|)$ stationary processes
  3. $v(t) + v(s) - v(|t-s|)$ stationary-increment processes
  4. $R(s,u) = R(s,t)R(t,u)/R(t,t)$, $s < t < u$, Markov processes
  5. $R(at, as) = a^{2H} R(t,s)$, $a > 0$, H-selfsimilar processes
- The Brownian motion has $R(t,s) = t \wedge s$.

$\Longrightarrow$ (1), (3), (4), (5) and (6) (with $H = \frac{1}{2}$) above hold.

$\Longrightarrow$

☺

Outline

Theorem (Fredholm Representation)

Let $X = (X_t)_{t\in[0,T]}$ be a separable centered Gaussian process. Then there exists a kernel $K_T \in L^2([0,T]^2)$ and a Brownian motion $W = (W_t)_{t\geq 0}$, such that

$$X_t = \int_0^T K_T(t,s)\,\mathrm{d}W_s$$

if and only if the covariance R of X satisfies the trace condition

$$\int_0^T R(t,t)\,\mathrm{d}t < \infty.$$

- The Fredholm Kernel K_T usually depends on T even if R does not.
- K_T may be assumed to be symmetric.
- K_T is unique in the sense that if there is another representation with kernel $\tilde{K}_T$, then $\tilde{K}_T = UK_T$ for some unitary operator U on $L^2([0,T])$. (And some other Brownian motion $\tilde{W}$.)
- The Fredholm Representation Theorem holds also for the parameter space $\mathbb{R}_+$, but the trace condition seldom holds, i.e. typically

$$\int_0^\infty R(t,t)\,\mathrm{d}t = \infty.$$

- If the covariance R is degenerate, one needs to extend the probability space to carry the Brownian motion.

- K_T (operator) can be constructed from R_T (operator) as the unique positive symmetric square-root, i.e. the operator K_T is a limit of polynomials:

$$K_T = \lim_{n\to\infty} P_n(R_T).$$

- The positive symmetric square-root is different from the Cholesky square-root. Indeed, the Cholesky square-root would correspond the Volterra Representation theorem

$$X_t = \int_0^t K_T(t,s)\,\mathrm{d}s.$$

  The Volterra representation does not hold for Gaussian processes in general.

Consider a truncated series expansion

$$X^n_t = \sum_{k=1}^{n} \int_0^t e_k^T(s)\,\mathrm{d}s \cdot \xi_k,$$

where (ξ_k) is i.i.d. $\sim \mathcal{N}(0,1)$ sequence, and (e_k^T) is an orthonormal basis in $L^2([0,T])$.

X^n is not *purely non-deterministic* ($\mathcal{F}_{0+}^{X^n}$ is not trivial) $\implies X$ cannot have Volterra representation.

On the other hand, by choosing (e_k^T) to be the trigonometric basis on $L^2([0,T])$, X^n is a finite-rank approximation of the Karhunen–Loève representation of standard Brownian motion on $[0,T]$.

$\implies$ as $n \to \infty$, $X^n \to W$ (a Volterra process, obviously).

Let B be the Brownian bridge, i.e., formally $B_t = W_t | \{W_T = 0\}$.

The orthogonal representation of B is

$$B_t = W_t - \frac{t}{T} W_T.$$

$\Longrightarrow$ B has a Fredholm representation with kernel

$$K_T(t,s) = \mathbf{1}_{[0,t)}(s) - \frac{t}{T}.$$

The canonical representation of the Brownian bridge is

$$B_t = (T-t) \int_0^t \frac{1}{T-s} \, \mathrm{d}W_s.$$

$\Longrightarrow$ B has also a Volterra representation with kernel

$$K_T(t,s) = \frac{T-t}{T-s}.$$

The Mercer's theorem (a.k.a. the eigenvalue decomposition) $\Longrightarrow$

$$R(t,s) = \sum_{i=1}^{\infty} \lambda_i^T e_i^T(t) e_i^T(s),$$

where (λ_i^T) and (e_i^T) are the eigenvalues and the eigenfunctions of the covariance operator

$$R_T f(t) = \int_0^T f(s) R(t,s)\,\mathrm{d}s,$$

and (e_i^T) is an orthonormal basis on $L^2([0,T])$.

R_T is a covariance operator $\Longrightarrow$ R_T admits a square-root operator K_T. The trace condition $\Longrightarrow$ R_T is trace-class $\Longrightarrow$ K_T is Hilbert–Schmidt.

$\Longrightarrow$ K_T admits a Kernel.

Indeed,

$$K_T(t,s) = \sum_{i=1}^{\infty} \sqrt{\lambda_i^T}\, e_i^T(t) e_i^T(s).$$

Now K_T is obviously symmetric and we have (by simple calculations)

$$R(t,s) = \int_0^T K_T(t,u) K_T(s,u)\, \mathrm{d}u.$$

$\Longrightarrow$ the Fredholm Representation follows (in law), and ω-by-ω by enlarging the probability space, if needed (details omitted, just trust me). □

Outline

The Wiener integral of a function f w.r.t. X, denoted by $\int_0^T f(t)\,\mathrm{d}X_t$, extends linearly the relation

$$X_t - X_s = \int_0^T \mathbf{1}_{[s,t)}(u)\,\mathrm{d}W_u,$$

closed under a suitable norm.

The Malliavin derivative of the random variable $\int_0^T f(t)\,\mathrm{d}X_t$ is the process

$$D_s \int_0^T f(t)\,\mathrm{d}X_t = f(s), \quad s \in [0, T].$$

Extend this under a suitable norm and obey the chain rule.

The Skorohod integral is the adjoint of the Malliavin derivative. It extends both the Wiener and the Itô integrals.

Outline

The adjoint operator Γ^* of a kernel $\Gamma \in L^2([0, T]^2)$ is defined by linearly extending the relation

$$\Gamma^* \mathbf{1}_{[0,t)} = \Gamma(t, \cdot).$$

This is not the same adjoint as in the Malliavin–Skorohod case before.

Remark

If $\Gamma(\cdot, s)$ is of bounded variation for all s and f is nice enough, then

$$\Gamma^* f(s) = \int_0^T f(t)\,\Gamma(\mathrm{d}t, s).$$

Theorem (Transfer Principle)

Let X be a Gaussian Fredholm process with kernel K_T. Let D_T, δ_T, D_T^W and δ_T^W be the Malliavin derivative and the Skorohod integral with respect to X and to the Brownian motion W. Then

$$\delta_T = \delta_T^W K_T^* \quad \text{and} \quad K_T^* D_T = D_T^W.$$

Proof: Trivial. □

Theorem (Itô Formula)

Let X be centered Gaussian process with covariance R and let $f \in C^2$. Then

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\, \delta_t X_s + \frac{1}{2}\int_0^t f''(X_s)\, \mathrm{d}R(s,s),$$

if anything.

Proof: Either trivial, straightforward or extremely technical, depending on the generality required. □

Outline

Recall the Hitsuda Representation Theorem: A centered Gaussian process $\tilde{W}$ is equivalent to a Brownian motion W if and only if there exists a Volterra kernel $\ell \in L^2([0,T]^2)$ such that

$$\mathrm{d}\tilde{W}_t = \mathrm{d}W_t + \int_0^t \ell(t,s)\,\mathrm{d}W_s \cdot \mathrm{d}t.$$

Now, let $\tilde{X}$ and X be Gaussian Fredholm processes with

$$\begin{aligned} \tilde{X}_t &= \int_0^T \tilde{K}_T(t,s)\,\mathrm{d}W_s, \\ X_t &= \int_0^T K_T(t,s)\,\mathrm{d}W_s. \end{aligned}$$

Suppose then that $\tilde{X}$ has (also) representation

$$\tilde{X}_t = \int_0^T K_T(t,s)\,\mathrm{d}\tilde{W}_s$$

where $\tilde{W}$ and W are equivalent.

Then, obviously $\tilde{X}$ and X are equivalent. By plugging in the Hitsuda connection we obtain

$$\tilde{X}_t = \int_0^T \left[K_T(t,s) + \int_s^T K_T(t,u)\ell(u,s)\,\mathrm{d}u \right] \mathrm{d}W_s.$$

Thus, we have shown the following:

Theorem (Equivalence of Laws)

Let X and $\tilde{X}$ be two Gaussian process with Fredholm kernels K_T and $\tilde{K}_T$, respectively. If there exists a Volterra kernel $\ell \in L^2([0,T]^2)$ such that

$$\tilde{K}_T(t,s) = K_T(t,s) + \int_s^T K_T(t,u)\ell(u,s)\,\mathrm{d}u,$$

then X and $\tilde{X}$ are equivalent in law.

If the kernel K_T satisfies an appropriate non-degeneracy property, then the condition above is also necessary.

In the same way, as in the case of equivalence of laws, we see that:

THEOREM (SERIES REPRESENTATION)

Let X be a Gaussian Fredholm process with kernel K_T and let (φ_k^T) be any orthonormal basis in $L^2([0,T])$. Then

$$X_t = \sum_{k=1}^{\infty} \int_0^T K_T(t,s)\varphi_k^T(s)\,\mathrm{d}s \cdot \xi_k,$$

where (ξ_k) is i.i.d. sequence of standard Gaussian random variables.

The series above converges in $L^2(\Omega)$; and also almost surely uniformly if and only if X is continuous.

Outline

References

- Sottinen, T. and Viitasaari, L. (2014) *Stochastic Analysis of Gaussian Processes via Fredholm Representation.* Preprint, arXiv:1410.2230.
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Thank you for listening!

Any questions?