Representations of Gaussian bridges

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Brownian bridge

Let \( o = (0, \xi) \to (T, \theta) \)

\[
dW_t^o = dW_t + \frac{\theta - W_t^o}{T-t} dt, \quad W_0^o = \xi, \quad (sde)
\]

\[
W_t^o = \xi + (\theta - \xi) \frac{t}{T} + (T-t) \int_0^t \frac{dW_s}{T-s},
\]

\[
W_t^o = \theta \frac{t}{T} + \left( W_t - \frac{t}{T} W_T \right).
\]

Here

\[
\text{Law}(W^o; P) = \text{Law}(W; P^o),
\]

\[
P^o = P(\cdot | W_T = \theta).
\]

Setting \( \theta = W_T^o \) we see that (sde) is the semimartingale decomposition of \( W^o \) in the the filtration \( \mathcal{F}_t^{W^o} \lor \sigma \{ W_T^o \} \) and \( W \) is a Brownian motion in this filtration.
General anticipative representation

Let $X$ be Gaussian with mean $\mu$ and covariance $R$. Then $X^o = X|X_T = \theta$ is Gaussian with with

$$E(X_t|X_T = \theta) = \frac{R(T, t)}{R(T, T)} (\theta - \mu(T)) + \mu(t),$$

$$\text{Cov}(X_t, X_s|X_T = \theta) = R(t, s) - \frac{R(T, t) R(T, s)}{R(T, T)}.$$

From the orthogonal decomposition of $X$ given $X_T$ we obtain an anticipative representation for any Gaussian bridge:

$$X_t^o = \theta \frac{R(T, t)}{R(T, T)} + \left( X_t - \frac{R(T, t)}{R(T, T)} X_T \right).$$
Abstract non-anticipative representation

Idea is to use the prediction martingale $m$ of $X$ and the Girsanov’s theorem.

Assumptions:

(A0) Filtration of $X$ is continuous

(A1) $P_t^o \sim P_t$ for all $t < T$.

(A2) The non-anticipative linear functional

$$F_T : X_t \mapsto m_t = \mathbf{E}(X_T|\mathcal{F}_t^X)$$

is injective
Abstract non-anticipative representation, cont.

Denote $\langle m \rangle_{T,t} := \langle m \rangle_T - \langle m \rangle_t$.

By using the Bayes’ rule and Itô’s formula we see that

$$dP_t^O = L_t^O dP_t,$$

where

$$\log L_t =$$

$$\int_0^t \frac{\theta - m_s}{\langle m \rangle_{T,s}} dm_s - \frac{1}{2} \int_0^t \left( \frac{\theta - m_s}{\langle m \rangle_{T,s}} \right)^2 d\langle m \rangle_s.$$

Let $S_o(m)$ be the solution of

$$dm_t^O = dm_t + \frac{\theta - m_t^O}{\langle m \rangle_{T,t}} d\langle m \rangle_t, \quad m_0^O = \zeta,$$

i.e.

$$m_t = S_o(m)_t =$$

$$\zeta + (\theta - \zeta) \frac{\langle m \rangle_t}{\langle m \rangle_T} + \langle m \rangle_{T,t} \int_0^t \frac{dm_s}{\langle m \rangle_{T,s}}.$$
Abstract non-anticipative representation, cont., cont.

By using the Girsanov’s theorem we see that if $X$ satisfies (A0), (A1) and (A2) then

$$X^o = F_T^{-1} \circ S_o \circ F_T(X).$$

This representation is non-anticipative.
Bridges of Gaussian martingales

Let $M$ be a continuous Gaussian martingale with strictly increasing bracket $\langle M \rangle$ and $M_0 = \xi$.

\[
    dM_t^O = dM_t + \frac{\theta - M_t^O}{\langle M \rangle_{T,t}} d\langle M \rangle_t, \quad M_0^O = \xi,
\]

\[
    M_t^O = \xi + (\theta - \xi)\frac{\langle M \rangle_t}{\langle M \rangle_T} + \langle M \rangle_{T,t} \int_0^t \frac{dM_s}{\langle M \rangle_{T,s}},
\]

\[
    M_t^O = \theta \frac{\langle M \rangle_t}{\langle M \rangle_T} + \left( M_t - \frac{\langle M \rangle_t}{\langle M \rangle_T} M_T \right).
\]

Moreover, we have

\[
    EM_t^O = \xi + (\theta - \xi)\frac{\langle M \rangle_t}{\langle M \rangle_T},
\]

\[
    \text{Cov}(M_t^O, M_s^O) = \langle M \rangle_{t \wedge s} - \frac{\langle M \rangle_t \langle M \rangle_s}{\langle M \rangle_T}.
\]

To see this just note that $R(t, s) = \langle M \rangle_{t \wedge s}$ and $F_T$ is, of course, the identity.
Bridges of Wiener predictable processes

Assume:

\[(A3) \quad m_t = \int_0^t p_T(t, s) dX_s,\]

\[(A4) \quad X_t = \int_0^t p_T^*(t, s) dm_s.\]

Given (A0) [cts. filtration], (A1) \([P_t^o \sim P_t]\), (A3) and (A4):

\[
X_t^o = X_t + \int_0^t \left\{ \theta - \int_0^s p_T(s, u) dX_u^o \right\} \frac{p_T^*(t, s)}{\langle m \rangle_{T,s}} d\langle m \rangle_s.
\]

\[
X_t^o = \theta \frac{R(T, t)}{R(T, T)} + X_t - \int_0^t \phi_T(t, s) dX_s,
\]

\[
\phi_T(t, s) = \\
\int_s^t \left\{ \int_s^u \frac{p_T(v, s)}{\langle m \rangle_{T,v}} d\langle m \rangle_v - \frac{p_T(u, s)}{\langle m \rangle_{T,u}} \right\} p_T^*(t, u) d\langle m \rangle_u.
\]
Bridges of Volterra processes

(A5) There exists a Volterra kernel $k$ and a continuous Gaussian martingale $M$ with strictly increasing bracket $\langle M \rangle$ such that

$$X_t = \int_0^t k(t, s) dM_s.$$

Let $K$ extend $1_{[0, t)} \mapsto k(t, \cdot)$ linearly and assume:

(A6) The equation $Kf = 1_{[0, t)}$ has a solution.

(A7) The equation $Kg = 1_{[0, t)}k(T, \cdot)$ has a solution.

By (A6) we may set $k^*(t, s) = K^{-1}1_{[0, t)}(s)$ we have

$$M_t = \int_0^t k^*(t, s) dX_s.$$

Since

$$dm_t = k(T, t) dM_t,$$

we have

$$X_t = \int_0^t \frac{k(t, s)}{k(T, s)} dm_s.$$
Bridges of Volterra processes, cont.

By (A7) we have

\[ m_t = X_t + \int_0^t \Psi_T(t, s) dX_s, \]

\[ \Psi_T(t, s) = \mathcal{K}^{-1} \left[ 1_{[0,t)} k(T, \cdot) \right](s). \]

So, we have found that

\[ d\langle m \rangle_t = k(T, t)^2 d\langle M \rangle_t, \]

\[ p_T(t, s) = 1_{[0,t)}(s) + \Psi_T(t, s), \]

\[ p_T^*(t, s) = \frac{k(t, s)}{k(T, s)}. \]

These functions are known explicitly if \( X \) is, for example, the fractional Brownian motion or the Riemann–Liouville process.