Parameter Estimation for the Langevin Equation with Stationary-Increment Gaussian Noise

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Abstract

We study the Langevin equation with stationary-increment Gaussian noise. We show the strong consistency and the asymptotic normality with Berry–Esseen bound of the so-called alternative estimator (AE) of the mean reversion parameter. The conditions and results are stated in terms of the variance function of the noise. We consider both the case of continuous and discrete observations.

As examples we consider fractional and bifractional Ornstein–Uhlenbeck processes.

We discuss the maximum likelihood and the least squares estimators.

This is joint work with Lauri Viitasaari (Aalto University, Finland).

Outline

Outline

We consider statistical parameter estimation for the unknown parameter $\theta > 0$ in the (generalized) Langevin equation

$$\begin{aligned} \mathrm{d}U_t^{\theta,\xi} &= -\theta U_t^{\theta,\xi}\,\mathrm{d}t + \mathrm{d}G_t, \qquad t \geq 0, \\ U_0^{\theta,\xi} &= \xi. \end{aligned}$$

The solution of the Langevin equation above is

$$U_t^{\theta,\xi} = \mathrm{e}^{-\theta t}\xi + \int_0^t \mathrm{e}^{-\theta(t-s)}\,\mathrm{d}G_s,$$

which can be seen by using the integration by parts.

Nothing was assumed here, except the finiteness of G!

Our estimation will be based on the solution that starts from zero. We denote $X^\theta = U^{\theta,0}$.

For the existence of the STATIONARY SOLUTION, G must have STATIONARY INCREMENTS. By extending G to the negative half-line with an independent copy, the stationary solution is

$$U_t^\theta = \int_{-\infty}^t \mathrm{e}^{-\theta(t-s)}\,\mathrm{d}G_s, \quad t \geq 0.$$

In other words, the stationary solution is $U^\theta = U^{\theta,\xi_{\mathrm{stat}}}$, with

$$\xi_{\mathrm{stat}} = \int_{-\infty}^0 \mathrm{e}^{-\theta t}\,\mathrm{d}G_t.$$

In particular, the stationary solution exists if and only if the integral above converges (almost surely), and in this case

$$X_t^\theta = U_t^\theta - \mathrm{e}^{-\theta t}U_0^\theta.$$

Assume that the noise G is centered Gaussian process with stationary increments. Denote

$$v(t) = \mathbf{E}[G_t^2] \quad \text{and} \quad r_\theta(t) = \mathbf{E}[U_0^\theta U_t^\theta].$$

Then

$$\begin{aligned} r_\theta(t) &= \theta \int_{-\infty}^0 \mathrm{e}^{\theta s} g(t,s)\,\mathrm{d}s \\ &\quad -\theta^2 \mathrm{e}^{-\theta t} \int_{-\infty}^t \int_{-\infty}^0 \mathrm{e}^{\theta(s+u)} g(s,u)\,\mathrm{d}s\mathrm{d}u, \end{aligned}$$

where

$$g(t,s) = \frac{1}{2}\Big[v(t) + v(s) - v(t-s)\Big]$$

is the covariance of G.

The following assumption ensures the EXISTENCE of the AE

Assumption (E)

The variance function v is strictly increasing.

The following assumption ensures the strong CONSISTENCY of the AE

Assumption (C)

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T |r_\theta(t)| \, \mathrm{d}t = 0.$$

Denote

$$w_\theta(T) = \frac{2}{T^2}\int_0^T\int_0^T r_\theta(t-s)^2 \,\mathrm{d}s\mathrm{d}t,$$
$$R_\theta(T) = \frac{\int_0^T |r_\theta(t)|\,\mathrm{d}t}{T\sqrt{w_\theta(T)}}.$$

- w_θ is the asymptiotic variance.
- R_θ is the Berry–Esseen bound.

For the asymptotic Normality of the AE, we assume

Assumption (N)

$$\lim_{T\to\infty} R_\theta(T) = 0.$$

Outline

If (E) holds, then $\theta \mapsto r_\theta(0)$ is invertible, and we can define:

Definition (Alternative Estimator)

The alternative estimator (AE) is

$$\tilde{\theta}_T = \psi^{-1}\left(\frac{1}{T}\int_0^T (X_t^\theta)^2 \,\mathrm{d}t\right),$$

where

$$\psi(\theta) = r_\theta(0) = \frac{\theta}{2}\int_0^\infty \mathrm{e}^{-\theta t} v(t)\,\mathrm{d}t$$

is the variance of the stationary solution.

If, in addition to (E), (C) holds, then U^θ is ergodic, and we obtain:

Theorem (Strong Consistency)

Suppose (E) and (C) hold. Then $\tilde{\theta}_T \to \theta$ almost surely.

Remark

Theorem (Strong Consistency) hold also without Gaussianity, if Assumption (C) is replaced by a suitable assumption ensuring the ergodicity of the stationary solution U^θ.

If (E), (C) and (N) hold, then we can invoke the Fourth Moment Theorem, and we obtain

Theorem (Asymptotic Normality)

Suppose (E) and (C) hold. Then for all $K > 0$,

$$\sup_{|x| \leq K} \left| \mathbf{P}\left[\frac{|\psi'(\theta)|}{\sqrt{w_\theta(T)}} \left(\tilde{\theta}_T - \theta \right) \leq x \right] - \Phi(x) \right| \leq C_{\theta,K} R_\theta(T).$$

In particular, if (N) holds, then

$$\frac{|\psi'(\theta)|}{\sqrt{w_\theta(T)}} \left(\tilde{\theta}_T - \theta \right) \xrightarrow{\mathrm{d}} \mathcal{N}(0,1).$$

Corollary (Classical Rate)

Suppose (E) holds. Assume $\int_0^\infty r_\theta(t)^2 \, \mathrm{d}t < \infty$. *Then* $\tilde{\theta}_T$ *is asymptotically normal with rate* $\sqrt{T}$.

Corollary (Mixed Models)

Let G^i*'s be independent, each satisfying (E) and (C), with* $r_{\theta,i} \geq 0$. *Then, for the mixed model*

$$\sup_{x \in [-K,K]} \left| \mathbf{P}\left[\frac{|\psi'(\theta)|}{\sqrt{w_\theta(T)}} \left(\tilde{\theta}_T - \theta \right) \leq x \right] - \Phi(x) \right|$$

$$\leq \quad C_{\theta,K} \max_{i=1,\ldots,n} \frac{\int_0^T r_{\theta,i}(t) \, \mathrm{d}t}{\sqrt{\int_0^T \int_0^T r_{\theta,i}(t-s)^2 \, \mathrm{d}s \mathrm{d}t}}.$$

In practice continuous observations are rarely available. Therefore, it is important to consider the case of discrete observations. To control the error introduced by the unobserved time-points, we assume that the driving noise G is Hölder continuous with some index $H \in (0,1)$. The general idea is, that the smaller the H the more care must be taken in choosing the time-mesh of the observations. The following assumption is necessary and sufficient for the Hölder continuity.

Assumption (H)

Let $H \in (0,1)$. For all $\varepsilon > 0$ there exists a constant C_ε such that

$$v(t) \leq C_\varepsilon \, t^{2H-\varepsilon}.$$

Alternative Estimator

Discrete Observations

Let $t_k = k\Delta_N$, $k = 0, \ldots, N$. Denote $T_N = N\Delta_N$ and assume that $\Delta_N \to 0$ with $T_N \to \infty$. The AE based on the discrete observations is

$$\tilde{\theta}_N = \psi^{-1}\left(\frac{1}{T_N}\sum_{k=1}^{N}(X^{\theta}_{k\Delta_N})^2\Delta_N\right).$$

Assumption (M)

Assume that

$$N\Delta_N^{\beta} \to 0,$$

where

$$\beta = \beta(H) = \frac{2H + \frac{1}{2}}{H + \frac{1}{2}} - \delta$$

for some $\delta > 0$.

Theorem (Discrete Observations)

Suppose (E), (C), (H) and (M) hold. Then,

$$\tilde{\theta}_N \to \theta \quad a.s.$$

Moreover, if (N) holds, then

$$\frac{|\psi'(\theta)|}{\sqrt{w_\theta(T_N)}} \left(\tilde{\theta}_N - \theta\right) \xrightarrow{\mathrm{d}} \mathcal{N}(0,1).$$

Remark (Bounds for Discrete Asymptotic Normality)

Berry–Esseen type upper bounds for the asymptotic normality are possible, but complicated.

Outline

The fractional Brownian motion B^H with Hurst index $H \in (0, 1)$ is the stationary-increment Gaussian process with variance function

$$v_H(t) = t^{2H}.$$

The Hurst index H is both the index of self-similarity and the Hölder continuity.

The fractional Ornstein–Uhlenbeck process (of the first kind) is the stationary solution to the Langevin equation

$$\mathrm{d}U_t^{H,\theta} = -\theta U_t^{H,\theta}\,\mathrm{d}t + \mathrm{d}B_t^H, \quad t \geq 0.$$

We have

$$r_{H,\theta}(t) \sim \frac{H(2H-1)}{\theta^2} t^{2H-2},$$
$$\psi_H(\theta) = \frac{H\Gamma(2H)}{\theta^{2H}}.$$

Consequently, (E), (C) and (H) are satisfied for all H, and (N) is satisfied for $H \leq 3/4$.

For $H < 3/4$,

$$\int_0^\infty r_{H,\theta}(t)^2 \,\mathrm{d}t = \theta^{-2H} \sigma_H^2,$$

where we have denoted

$$\sigma_H^2 = \int_0^\infty r_{H,1}(t)^2 \,\mathrm{d}t.$$

Proposition (Fractional Ornstein–Uhlenbeck Process of the First Kind)

1. *Let* $H \in (0, 1/2]$. *Then* $\sup_{x\in[-K,K]} \left| \mathbf{P}\left[\sqrt{\frac{T}{\theta\sigma_H^2}} \left(\tilde{\theta}_T^H - \theta \right) \le x \right] - \Phi(x) \right| \le \frac{C_{H,\theta,K}}{\sqrt{T}}$.
2. *Let* $H \in (1/2, 3/4)$. *Then* $\sup_{x\in[-K,K]} \left| \mathbf{P}\left[\sqrt{\frac{T}{\theta\sigma_H^2}} \left(\tilde{\theta}_T^H - \theta \right) \le x \right] - \Phi(x) \right| \le \frac{C_{H,\theta,K}}{\sqrt{T^{3-4H}}}$.
3. *Let* $H = 3/4$. *Then* $\sup_{x\in[-K,K]} \left| \mathbf{P}\left[\sqrt{\frac{T}{\theta\sigma^2 \log T}} \left(\tilde{\theta}_T^{3/4} - \theta \right) \le x \right] - \Phi(x) \right| \le \frac{C_{3/4,\theta,K}}{\sqrt{\log T}}$.

The fractional Ornstein–Uhlenbeck process of the second kind is the stationary solution of the Langevin equation with the noise

$$G_t^H = \int_0^t \mathrm{e}^{-s} \mathrm{d}B^H_{H\mathrm{e}^{s/H}}.$$

The idea of the construction above is to use the self-similarity of the fractional Brownian motion and the associated Lamperti transform (a.k.a. Doob transform).

For Brownian motion the Ornstein–Uhlenbeck processes of the first and second kind are the same. In general they are different.

The autocovariance $r_{H,\theta}$ of the fractional Ornstein–Uhlenbeck process of the second kind has EXPONENTIAL DECAY. Therefore, we have the following:

Proposition (Fractional Ornstein–Uhlenbeck Process of the Second Kind)

$$\sup_{x\in[-K,K]} \left| \mathbf{P}\left[\frac{\sqrt{T}}{\sigma_{H,\theta}(\theta)} \left(\tilde{\theta}_T^H - \theta \right) \leq x \right] - \Phi(x) \right| \leq \frac{C_{H,\theta,K}}{\sqrt{T}},$$

where

$$\sigma^2_{H,\theta}(\theta) = 4 \int_0^\infty r_{H,\theta}(t)^2 \, \mathrm{d}t.$$

The bifractional Brownian motion $B^{H,K}$ with parameters $H \in (0,1)$ and $K \in (0,1]$ is the Gaussian process with covariance

$$\mathbf{E}\Big[B_t^{H,K} B_t^{H,K}\Big] = \frac{1}{2^K}\left[(t^{2H}+s^{2H})^K - |t-s|^{2HK}\right].$$

It does not have stationary increments, except for $K = 1$. Consequently, there is no way to define the bifractional Ornstein–Uhlenbeck process of the first kind that would have a stationary version.

The bifractional Brownian motion is HK-self-similar. Consequently, we can define the bifractional Ornstein–Uhlenbeck process by replacing B^{HK} with $B^{H,K}$ in the case for fractional Ornstein–Uhlenbeck process of the second kind.

The autocovariance $r_{H,K,\theta}$ of the bifractional Ornstein–Uhlenbeck process of the second kind has exponential decay. Therefore:

Proposition (The bifractional Ornstein–Uhlenbeck processes of the second Kind)

$$\sup_{x\in[-L,L]} \left| \mathbf{P}\left[\frac{\sqrt{T}}{\sigma_{H,K,\theta}(\theta)} \left(\tilde{\theta}_T^{H,K,\theta} - \theta \right) \le x \right] - \Phi(x) \right| \le \frac{C_{H,K,\theta,L}}{\sqrt{T}},$$

where

$$\sigma^2_{H,K,\theta}(\theta) = 4 \int_0^\infty r_{H,K,\theta}(t)^2 \,\mathrm{d}t.$$

Outline

One the one hand, the LSE

$$\hat{\theta}_T = -\frac{\int_0^T X_t^\theta \, \delta X_t^\theta}{\int_0^T (X_t^\theta)^2 \, \mathrm{d}t}$$

arises heuristically by minimizing

$$\int_0^T |\dot{X}_t^\theta + \theta X_t^\theta|^2 \, \mathrm{d}t.$$

On the other hand, one would hope that

$$\int_0^T X_t^\theta \, \delta X_t^\theta = -\theta \int_0^T (X_t^\theta)^2 \, \mathrm{d}t + \int_0^T X_t^\theta \, \delta G_t.$$

This would lead to the LSE

$$\widehat{\theta}_T = \theta - \frac{\int_0^T X_t^\theta \, \delta G_t}{\int_0^T (X_t^\theta)^2 \, \mathrm{d}t}.$$

Discussion on Other Estimators

Least Squares Estimator (LSE)

Unfortunately, the Skorohod integral is not (bi)linear. In particular, the equation for it in the previous slide does not hold. Consequently, $\hat{\theta}_T$ and $\widehat{\theta}_T$ not the same.

$\widehat{\theta}_T$ has been shown to be consistent for some fractional Ornstein–Uhlenbeck processes. However, the $\widehat{\theta}_T$ depends on θ, the parameter we want to estimate! $\hat{\theta}_T$ is even worse: it will fail under rather general assumptions:

Proposition ($\hat{\theta}_T$ Failure)

Assume that U^θ is ergodic. If $(X_T^\theta)^2/T \to 0$ in $L^1(\Omega)$ and almost surely, then

$$\hat{\theta}_T \to 0 \quad a.s.$$

Assume the following (Inverse) Volterra representation: There exists a Gaussian martingale M with bracket $\langle M \rangle$ and a kernel $k \in L^2_{\mathrm{loc}}(\mathbb{R}^2_+, \mathrm{d}\langle M \rangle \times \mathrm{d}\langle M \rangle)$ such that

$$
\begin{aligned}
G_t &= \int_0^t k(t,s)\,\mathrm{d}M_s, \\
M_t &= \int_0^t k^*(t,s)\,\mathrm{d}G_t.
\end{aligned}
$$

The authors have very little idea, when such representations exist!

Moreover, assume the existence of

$$
\begin{aligned}
M^\theta_t &= \int_0^t k^*(t,s)\,\mathrm{d}X^\theta_t, \\
\Xi^\theta_t &= \frac{\mathrm{d}}{\mathrm{d}\langle M \rangle_t} \int_0^t k^*(t,s) X^\theta_s\,\mathrm{d}s.
\end{aligned}
$$

Proposition (MLE)

Assume that $\Xi^\theta \in L^2(\Omega \times [0, T], \mathrm{d}\mathbf{P} \times \mathrm{d}\langle M \rangle)$. *Then the MLE based on the observations* X_t^θ, $t \in [0, T]$, *is*

$$\bar{\theta}_T = -\frac{\int_0^T \Xi_t^\theta \, \mathrm{d}M_t^\theta}{\int_0^T (\Xi_t^\theta)^2 \, \mathrm{d}\langle M \rangle_t}.$$

Moreover, if $\int_0^T (\Xi_t^\theta)^2 \, \mathrm{d}\langle M \rangle_t \to \infty$ *almost surely, then the MLE is strongly consistent.*

Thank you for listening!
Any questions?