Completely Correlated Mixed Fractional Brownian Motion

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The Reference

Josephine Dufitinema, Foad Shokrollahi, Tommi Sottinen, Lauri Viitasaari (2021)
Long-range dependent completely correlated mixed fractional Brownian motion,
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Abstract

We introduce the long-range dependent completely correlated mixed fractional Brownian motion (ccmfBm). This is a process that is driven by a mixture of Brownian motion (Bm) and a long-range dependent completely correlated fractional Brownian motion (fBm, ccfBm) that is constructed from the Brownian motion via the Molchan–Golosov representation. Thus, there is a single Bm driving the mixed process. In the short time-scales the ccmfBm behaves like the Bm (it has Brownian Hölder index and quadratic variation). However, in the long time-scales it behaves like the fBm (it has long-range dependence governed by the fBms Hurst index). We provide a transfer principle for the ccmfBm and use it to construct the Cameron–Martin–Girsanov–Hitsuda theorem and prediction formulas. Finally, we illustrate the ccmfBm by simulations.

Outline

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Construction of ccmfBm

Take a Brownian motion (Bm) $W = (W_t)_{t\in[0,T]}$. Construct a completely correlated fractional Brownian motion (ccfBm, fBm) with $H > 1/2$ from the Bm by using the Molchan–Golosov kernel

$$\begin{aligned} B_t^H &= \int_0^t K_H(t,s)\,\mathrm{d}W_s, \\ K_H(t,s) &= c(H)\frac{1}{s^{H-\frac{1}{2}}}\int_s^t \frac{u^{H-\frac{1}{2}}}{(u-s)^{\frac{3}{2}-H}}\,\mathrm{d}u, \end{aligned}$$

and then, from the SAME Bm construct the long-range dependent completely correlated mixed fractional Brownian motion (ccmfBm):

$$X_t = X_t^{a,b,H} = aW_t + bB_t^H.$$

Outline

Motivation

The ccmfBm does not have stationary increments. A more natural mixed fractional Brownian motion (mfBm) would be

$$M_t = aW_t + bB_t^H,$$

where W and B^H are independent. This process has been studied in many articles.

However, ccmfBm is more convenient than mfBm because, as we will see, it has easier INVERSE TRANSFER PRINCIPLE. Also, the ccmfBm and the mfBm are similar in the sense that their short-time and long-time behaviors are mostly same (Hölder continuity, quadratic variation, long-range dependence).

Outline

(Inverse) Transfer Principle

Let $L^2 = L^2([0,T])$. For a kernel $K\colon [0,T]^2 \to \mathbb{R}$ its associated operator is

$$\mathrm{K}f(t) = \int_0^T f(s)K(t,s)\,\mathrm{d}u.$$

The adjoint associated operator K^* of a kernel K is defined by linearly extending the relation

$$\mathrm{K}^*\mathbf{1}_t(s) = K(t,s),$$

where $\mathbf{1}_t = \mathbf{1}_{[0,t)}$ is the indicator function.

(Inverse) Transfer Principle

If $K(\cdot, t)$ has bounded variation, then (more or less)

$$\mathrm{K}^* f(t) = f(t)K(T,t) + \int_t^T [f(u) - f(t)]\, K(\mathrm{d}u, t).$$

Since the Molchen–Golosov kernel $K_H(t,s)$ for $H > 1/2$ is differentiable in t and $K_H(t, t-) = 0$, its adjoint associated operator can be written as

$$\mathrm{K}_H^* f(t) = \int_t^T f(u) \frac{\partial K_H}{\partial u}(u,t)\, \mathrm{d}u.$$

(Inverse) Transfer Principle

Let Λ be the closure of the indicator functions $\mathbf{1}_t$, $t \in [0, T]$, under the inner product generated by the relation

$$\langle \mathbf{1}_t, \mathbf{1}_s \rangle_\Lambda = R(t, s),$$

where R is the covariance of the ccmfBm.

Let $\mathcal{H}_1$ be the linear space, or first chaos, of X, i.e., the closure of the random variables X_t, $t \in [0, T]$, in $L^2(\Omega)$.

For $f \in \Lambda$ the abstract Wiener integral

$$\int_0^T f(t)\,\mathrm{d}X_t$$

is the image of the isometry $\mathbf{1}_t \mapsto X_t$ from Λ to $\mathcal{H}_1$.

Denote $L(t, s) = a\mathbf{1}_t(s) + bK_H(t, s)$ and let L and L^* be the associated and adjoint associated operators of L.

(Inverse) Transfer Principle

Lemma (1)

L^* *is a bounded operator on* L^2 *and it can be represented as*

$$\begin{aligned}\mathrm{L}^* f(t) &= af(t) + b\int_t^T f(u)\frac{\partial K_H}{\partial u}(u,t)\,\mathrm{d}u \\ &= af(t) + \frac{bc(H)}{t^{H-\frac{1}{2}}}\int_t^T f(u)\frac{u^{H-\frac{1}{2}}}{(u-t)^{\frac{3}{2}-H}}\,\mathrm{d}u.\end{aligned}$$

(Inverse) Transfer Principle

Beef of Proof: K_H^* is bounded on L^2, because

$$\begin{aligned}
\|\mathrm{K}_H^* f\|_2^2 &= \int_0^T \int_0^T f(t)f(s)\frac{\partial^2 R_H}{\partial s \partial t}(t,s)\,\mathrm{d}s\mathrm{d}t \\
&= H(2H-1)\int_0^T \int_0^T \frac{f(t)f(s)}{|t-s|^{2-2H}}\,\mathrm{d}s\mathrm{d}t \\
&\leq H(2H-1)\int_0^T \int_0^T \frac{f(t)^2}{|t-s|^{2-2H}}\,\mathrm{d}s\mathrm{d}t \\
&\leq H(2H-1)\frac{T^{2H-1}}{H-\frac{1}{2}}\|f\|_2^2,
\end{aligned}$$

where we have used the elementary estimate

$$2|f(t)f(s)| \leq f(t)^2 + f(s)^2$$

and symmetry.

(Inverse) Transfer Principle

Lemma (2)

For each $t \in [0, T]$, the integral equation

$$\mathbf{1}_t(s) = aL^{-1}(t,s) + b\int_s^T L^{-1}(t,u)\frac{\partial K_H}{\partial u}(u,s)\,\mathrm{d}u$$

admits the unique L^2-solution given by

$$L^{-1}(t,s) = \frac{1}{a}\mathbf{1}_t(s) + \frac{1}{a}\sum_{k=1}^{\infty}(-1)^k\left(\frac{b}{a}\right)^k \gamma_k(t,s)$$

where

$$\gamma_k(t,s) = \frac{c(H)^k\Gamma(H-\frac{1}{2})^k}{\Gamma\left(k\left(H-\frac{1}{2}\right)\right)}\frac{1}{s^{H-\frac{1}{2}}}\int_s^t u^{H-\frac{1}{2}}(u-s)^{k(H-\frac{1}{2})-1}\,\mathrm{d}u.$$

(Inverse) Transfer Principle

Beef of Proof: Denote

$$G(s,u) = -\frac{bc(H)}{a}\frac{u^{H-\frac{1}{2}}}{s^{H-\frac{1}{2}}(u-s)^{\frac{3}{2}-H}}.$$

Then Lemma 2 has the anti-Volterra equation of the second kind

$$\frac{1}{a}\mathbf{1}_t(s) = L^{-1}(t,s) - \int_s^t L^{-1}(t,u)G(s,u)\,\mathrm{d}u.$$

Lemma 1 implies that the solution of the equation in Lemma 2 is

$$L^{-1}(t,s) = \sum_{k=1}^{\infty} \mathrm{G}^k\left[\frac{1}{a}\mathbf{1}_t\right](s),$$

where G^0 is the identity operator and $\mathrm{G}^{k+1} = \mathrm{G}\mathrm{G}^k$. Finally, we use induction with the formula ($\alpha = H - \frac{1}{2}$)

$$\int_s^u (v-s)^{k\alpha-1}(u-v)^{\alpha-1}\,\mathrm{d}v = \frac{\Gamma(k\alpha)\Gamma(\alpha)}{\Gamma((k+1)\alpha)}(u-s)^{(k+1)\alpha-1}.$$

(Inverse) Transfer Principle

Theorem (1)

The ccmfBm X is an invertible Gaussian Volterra process in the sense that the process W defined as the abstract Wiener integral

$$W_t = \int_0^t L^{-1}(t,s)\,\mathrm{d}X_s$$

is the Bm from which the ccmfBm is constructed:

$$X_t = \int_0^t L(t,s)\,\mathrm{d}W_s.$$

(Inverse) Transfer Principle

Theorem ((Inverse) Transfer Principle)

Let $f \in L^2$. Let X be the ccmfBm constructed from the Bm W. Then

$$
\begin{aligned}
\int_0^T f(t)\,\mathrm{d}X_t &= \int_0^T \mathrm{L}^* f(t)\,\mathrm{d}W_t, \\
\int_0^T f(t)\,\mathrm{d}W_t &= \int_0^T (\mathrm{L}^*)^{-1} f(t)\,\mathrm{d}X_t,
\end{aligned}
$$

where

$$
\begin{aligned}
\mathrm{L}^* f(t) &= af(t) + b\int_t^T f(s)\frac{\partial K_H}{\partial s}(s,t)\,\mathrm{d}s, \\
(\mathrm{L}^*)^{-1} f(t) &= f(t)L^{-1}(T,t) + \int_t^T [f(s) - f(t)]\, L^{-1}(\mathrm{d}s, t).
\end{aligned}
$$

Outline

Applications

- Cameron–Martin–Girsanov–Hitsuda theorem, equivalence of laws
- Maximum likelihood estimation
- Prediction laws, bridges, conditional laws
- Simulation
- Malliavin calculus
- ...

Denote

$$\Psi(t, s|u) = (\mathrm{L}^*)^{-1}[L(t, \cdot) - L(u, \cdot)](s)$$

Then

$$\begin{aligned}
\hat{m}_t^X(u) &= \mathbf{E}\left[X_t \middle| \mathcal{F}_u^X\right] \\
&= \mathbf{E}\left[\int_0^t L(t, s)\,\mathrm{d}W_s \middle| \mathcal{F}_u^W\right] \\
&= \int_0^u L(t, s)\,\mathrm{d}W_s \\
&= \int_0^u L(u, s)\,\mathrm{d}W_s + \int_0^u \left[L(t, s) - L(u, s)\right]\,\mathrm{d}W_s \\
&= X_u - \int_0^u (\mathrm{L}^*)^{-1}\left[L(t, \cdot) - L(u, \cdot)\right](s)\,\mathrm{d}X_s. \\
&= X_u - \int_0^u \Psi(t, s|u)\,\mathrm{d}X_s
\end{aligned}$$

$$
\begin{aligned}
\hat{R}_X(t, s|u) &= \mathbf{E}\left[\left(X_t - \hat{m}_t^X(u)\right)\left(X_s - \hat{m}_s^X(u)\right) \middle| \mathcal{F}_u^X\right] \\
&= \mathbf{E}\left[\int_u^t L(t,x)\,\mathrm{d}W_x \int_u^s L(s,x)\,\mathrm{d}W_x \middle| \mathcal{F}_u^W\right] \\
&= \mathbf{E}\left[\int_u^t L(t,x)\,\mathrm{d}W_x \int_u^s L(s,x)\,\mathrm{d}W_x\right] \\
&= \int_u^{t\wedge s} L(t,x)L(s,x)\,\mathrm{d}x \\
&= \int_0^{t\wedge s} L(t,x)L(s,x)\,\mathrm{d}x - \int_0^u L(t,x)L(s,x)\,\mathrm{d}x \\
&= R(t,s) - \int_0^u L(t,x)L(s,x)\,\mathrm{d}x.
\end{aligned}
$$

Thank you for listening!
Any questions?