Non-semimartingales in finance

Pricing and Hedging Options with Quadratic Variation

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Reasons against non-semimartingales:

- In stochastic finance one needs integration theory to define self-financing trading strategies.
- The semimartingale theory for integration is well-suited for stochastic finance.
- Also, since semimartingales are the largest class of integrators that have continuous integrals, one expects arbitrage with non-semimartingales.
- It is an economic axiom that there should be no arbitrage.

Reasons for non-semimartingales:

- Some stylized facts (e.g. long range dependence) are difficult to incorporate into the semimartingale world.
- Even with semimartingales one does not use actual probabilities (but the so-called equivalent martingale measure). So, the role of probability in stochastic finance is small, if it even exists. And semimartingale is a probabilistic concept.

Outline

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Let $(B_t)_{t \in [0,T]}$ be the bond, or bank account. We work in the discounted world:

$$B_t = 1 \qquad \text{for all } t \in [0, T].$$

The bank account is riskless, i.e. non-random.

Let $S = (S_t)_{t \in [0,T]}$ be the stock. The stock is risky, i.e. random.

1. Classical assumption: S is a (continuous) semimartingale, typically the geometric Brownian motion.
2. Our assumption: S is continuous and has quadratic variation (We elaborate what we mean by quadratic variation later.)

Definition (Option)

Option is simply a real-valued mapping $S \mapsto G(S)$. The asset S is the underlying of the option G.

Example

- $G = (S_T - K)^+$ is a call-option,
- $G = (K - S_T)^+$ is a put-option,
- $G = S_T - K$ is a future.

T is the time of maturity and K is the strike-price.

DEFINITION (SELF-FINANCING TRADING STRATEGY)

TRADING STRATEGY $\Phi = (\Phi_t)_{t \in [0,T]}$ is an S-adapted stochastic process that tells the units of the underlying asset S the investor has is her portfolio at any time $t \in [0, T]$. The WEALTH of a SELF-FINANCING trading strategy Φ satisfies the FORWARD DIFFERENTIAL

$$\mathrm{d}V_t(\Phi) \;=\; \Phi_t \,\mathrm{d}S_t. \tag{1}$$

REMARK

(1) corresponds to the BUDGET CONSTRAINT

$$V_{t+\Delta t} \;=\; \Phi_t S_{t+\Delta t} + (V_t - \Phi_t S_t)\,.$$

Replication principle is used to hedge and price options.

Definition (Replication principle)

Let G be an option. Suppose that there is a trading strategy Φ with wealth $V_t(\Phi)$ at time t such that $G = V_T(\Phi)$ at time T. Then the price of the option G at time t is $V_t(\Phi)$.

The replication requirement $G = V_T(\Phi)$ can be written as

$$G = V_t(\Phi) + \int_t^T \Phi_s \,\mathrm{d}S_s,$$

where the integral is a forward integral (to be defined properly in the next section).

No-arbitrage principle can be used to price options.

Definition (Arbitrage)

Arbitrage is a self-financing trading strategy Φ such that $V_0(\Phi) = 0$, $\mathbf{P}[V_T(\Phi) \geq 0] = 1$, and $\mathbf{P}[V_T(\Phi) > 0] > 0$.

Definition (No-Arbitrage Principle)

No-Arbitrage Price of an option is any price that does not induce arbitrage into the market when the option is considered as a new asset.

REMARK

- If an option has a replication price then its no-arbitrage price must be the same. Otherwise one could make arbitrage by buying or selling the option with the no-arbitrage price and then replicating the option (or the "minus option"). The price difference would be arbitrage.
- It is possible in theory that an option has a replication price but no no-arbitrage prices. It this case the market already has arbitrage.
- It is also possible to have no-arbitrage prices, but no replication prices. In this case there are many no-arbitrage prices.

Outline

Definition (Forward Integral)

Let (Π_n) be a sequence of partitions of $[0, T]$ such that

$$\mathrm{mesh}(\Pi_n) = \sup_{t_k \in \Pi_n} |t_k - t_{k-1}| \to 0, \quad n \to \infty.$$

Forward integral of f w.r.t. g on $[0, T]$ is

$$\int_0^T f(t)\,\mathrm{d}g(t) = \lim_{n\to\infty} \sum_{t_k \in \Pi_n} f(t_{k-1})\,(g(t_k) - g(t_{k-1}))\,.$$

Forward integral of f w.r.t. g on $[a, b] \subset [0, T]$ is

$$\int_a^b f(t)\,\mathrm{d}g(t) = \int_0^T f(t)\mathbf{1}_{[a,b]}(t)\,\mathrm{d}g(t).$$

Remark

- The existence (and maybe even the value) of forward integral depends on the particular choice of the sequence of partitions.
- In what follows the sequence of partitions is assumed to be refining.
- The forward integral is a pathwise (almost sure) Ito integral if the sequence of partitions is chosen properly and the integrator is a semimartingale.
- In general case there is nothing the will a priori ensure the existence of the forward integral.
- We will see soon that if the integrator has quadratic variation then certain kind of forward integrals will exist.

Definition (Quadratic Variation)

Let (Π_n) be a sequence of (refining) partitions of $[0, T]$ such that $\mathrm{mesh}(\Pi_n) \to 0$. Quadratic variation of f on $[0, t]$ is

$$\langle f \rangle (t) = \lim_{n\to\infty} \sum_{t_k \in \Pi_n, t_k \le t} (f(t_k) - f(t_{k-1}))^2 .$$

Remark

Quadratic covariation can be defined in the same way or by using the polarization formula

$$\langle f, g \rangle = \frac{1}{4} (\langle f + g \rangle - \langle f - g \rangle) .$$

Lemma

Let f and g be continuous quadratic variation functions.

1. *For standard Brownian motion $\langle W \rangle_t = t$ a.s., if the sequence of partitions is refining.*
2. *If f is smooth then $\langle f \rangle = 0$.*
3. *If $\langle g \rangle = 0$ then $\langle f + g \rangle = \langle f \rangle$.*
4. *If f is smooth then*

$$\langle f \circ g \rangle (t) \;=\; \int_0^t f'\big(g(s)\big)^2 \,\mathrm{d}\langle g \rangle(s).$$

5. *If $\langle g \rangle = 0$ then $\langle f, g \rangle = 0$.*

Proof.

Item 1 This is a well-know fact from stochastic analysis (and too tedious to prove here).

Item 2 follows from the mean value theorem:

$$\begin{aligned} & \sum \left(f(t_k) - f(t_{k-1})\right)^2 \\ =: & \sum \left(\Delta f(t_k)\right)^2 \\ = & \sum \left(f'(\xi_k)\right)^2 (t_k - t_{k-1})^2 \\ \leq & 4\|f'\|_\infty^2 \mathrm{mesh}(\Pi_n) t. \end{aligned}$$

□

Proof.

Item 3 follows from Cauchy–Schwarz inequality:

$$\begin{aligned}
&\sum (\Delta(f+g))^2 \\
&\quad = \sum (\Delta f + \Delta g)^2 \\
&\quad = \sum (\Delta f)^2 + \sum (\Delta g)^2 + 2 \sum \Delta f \Delta g \\
&\quad \leq \sum (\Delta f)^2 + \sum (\Delta g)^2 + 2\sqrt{\sum (\Delta f)^2 \sum (\Delta g)^2}.
\end{aligned}$$

Item 4 follows from the mean value theorem:

$$(\Delta f(g(t_k)))^2 = f'(g(\xi_k))^2 (\Delta g(t_k))^2 .$$

Item 5 follows directly from the Cauchy–Schwarz inequality. □

Theorem (Ito's Formula)

Let $X = (X^1, \ldots, X^n)$ be a.s. continuous quadratic covariation process, and let $f : \mathbb{R}^n \to \mathbb{R}$ be smooth. Then, a.s.,

$$\mathrm{d}f(X_t) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(X_t)\mathrm{d}X_t + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x^i \partial x^j}(X_t)\,\mathrm{d}\left\langle X^i, X^j \right\rangle_t .$$

Proof.

Taylor is all you need! □

Remark

- The Ito formula implies that the forward integral exists and has a continuous modification.
- The Ito formula for quadratic variation processes is formally the same as the Ito formula for the continuous semimartingales.
- The differences between the forward world and the semimartingale worlds are:
  - The existence of the quadratic variation (limit in probability) is guaranteed for semimartingales.
  - In the forward world the integrals are always defined a.s.

Outline

Let W be the standard Brownian motion.

Definition (Black–Scholes Model)

In the Classical Black–Scholes model the stock price process is a geometric Brownian motion

$$S_t = S_0 e^{\mu t - \sigma^2/2t + \sigma W_t},$$

or as an Ito differential

$$\mathrm{d}S_t = S_t \left(\mu \, \mathrm{d}t + \sigma \, \mathrm{d}W_t\right).$$

Definition (Completeness)

A market model is complete if all options can be replicated with a self-financing trading strategy.

Theorem

The Black–Scholes market model is complete (at least for L^2-options).

Proof.

This follows from the martingale representation theorem. □

Theorem

The Black–Scholes market model is free of arbitrage (for tame strategies).

Proof.

This follows from the fact that Ito integrals are (proper) martingales for tame integrators. □

Remark

There is arbitrage in the Black–Scholes model with e.g. doubling strategies. (The proponents of semimartingale approach do not like to talk too much about it.)

Outline

Let $G = g(S_T)$ be a path-independent option. (We can consider more complicated options. We talk about them later in the final section.)

Assume that

$$\mathrm{d}\left\langle S\right\rangle_t \;=\; \sigma^2 S_t^2 \mathrm{d}t \tag{2}$$

Remark

- The classical Black–Scholes model satisfies (2).
- Let

$$\mathrm{d}S_t \;=\; S_t \left(\mu \mathrm{d}t + \sigma W_t + \nu Z_t\right),$$

  where Z is a fractional Brownian motion with $H > 1/2$. This model also satisfies (2), but now S is no longer a semimartingale.

By Ito formula, if $v(t,x)$ is the solution to Black-Scholes BPDE

$$\frac{\partial v}{\partial t}(t,x) - \frac{\sigma^2}{2}x^2\frac{\partial^2 v}{\partial x^2}(t,x) \;=\; 0$$

with boundary condition $v(T,x) = g(x)$, then

$$V_t(\Phi) \;=\; v(t,S_t)$$

and

$$\Phi_t \;=\; \frac{\partial v}{\partial x}(t,S_t).$$

Remark

Note that the prices and replications of (path-independent) options were derived from the quadratic variation property. Probability does not come into it!

Remark

It is true, by the Feynman–Kac formula, that

$$v(t,x) \;=\; \mathbf{E}\left[g\left(xe^{\sigma W_{T-t}-\frac{\sigma^2}{2}(T-t)}\right)\right],$$

where W is the standard Brownian motion. This does not imply that S is geometric Brownian motion of even log-normal!

Outline

- No-arbitrage and completeness are opposite requirements: the more you want to hedge the more sophisticated trading strategies you must allow. The more sophisticated trading strategies you allow the more candidates for arbitrage you have.
- In classical semimartingale theory completeness and arbitrage is attained in the class of either integrable enough or bounded from below strategies.
- In the quadratic variation world we would then like to identify a class of strategies that is
  1. Big enough to contain hedges for relevant options.
  2. Small enough to exclude arbitrage opportunities.
  3. Economically meaningful.

At this moment we propose the following class of SMOOTH STRATEGIES:

$$\Phi_t \;=\; \varphi(t, S_t, g^1(t,S), \ldots g^n(t,S)),$$

where φ is smooth and g^i's are HINDSIGHT FACTORS:

1. $g^i(\cdot, S)$ is S-adapted.
2. $g^i(\cdot, S)$ is continuous bounded variation process.
3. $\left|\int_0^t f(u)\mathrm{d}g^i(u,S) - \int_0^t f(u)\mathrm{d}g^i(u,\tilde{S})\right| \leq K\|f\|_\infty\|S-\tilde{S}\|_\infty.$

Definition (Stopping-Smooth)

Trading strategy is stopping-smooth if it is of the form

$$\Phi_t \;=\; \sum_{k=1}^{n} \Phi_t^k \mathbf{1}_{(\tau_k, \tau_{k+1}]}(t),$$

where Φ^k's are smooth and the stopping times τ_k are locally continuous.

A function f is locally continuous at point x if there exist an open set U such that $x \in \bar{U}$ and $f(x_n) \to f(x)$ whenever $x_n \in U$.

Theorem

There are no arbitrage opportunities in the class of stopping-smooth strategies if S has conditional full support:

$$\mathbf{P}\left[\sup_{s\in[t,T]} |S_t - \eta| \le \varepsilon \,\middle|\, \mathcal{F}_t^S\right] > 0$$

for all positive paths η s.t. $\eta(t) = S_t$.

References

1. Bender, C., Sottinen, T. and Valkeila, E. (2008) Pricing by hedging and no-arbitrage beyond semimartingales. *Finance and Stochastics* **12**, 441–468
2. Föllmer, H. (1980) Calcul d'Itô sans probabilités. *Seminar on Probability* **XV**.
3. Sondermann, D. (2006) *Introduction to stochastic calculus for finance. A new didactic approach.* Lecture Notes in Economics and Mathematical Systems **579**. Springer-Verlag, Berlin.