Black–Scholes Prices and Hedges for Financial Derivatives in Non-Gaussian Non-Martingale Models

Tommi Sottinen
University of Vaasa, Finland
http://www.uwasa.fi/∼tsottine/

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Outline

1. Market models, and self-financing strategies

- Let $\mathcal{C}_{s_0,+}$ be the space of continuous positive paths $\eta : [0, T] \to \mathbb{R}$ with $\eta(0) = s_0$.
  A discounted market model is five-tuple $(\Omega, \mathcal{F}, (S_t), (\mathcal{F}_t), \mathbf{P})$ where the stock-price process S takes values in $\mathcal{C}_{s_0,+}$.
- Non-anticipating trading strategy Φ is self-financing if its wealth satisfies

$$V_t(\Phi, v_0; S) = v_0 + \int_0^t \Phi_r \,\mathrm{d}S_r, \quad t \in [0, T]. \tag{1}$$

  Here the economic notion 'self-financing' is captured by the 'forward' construction of the pathwise integral in (1).

2. Pricing with replication, and arbitrage

- An OPTION is a mapping $G : \mathcal{C}_{s_0,+} \to \mathbb{R}_+$.
- The FAIR PRICE of an option G is the capital v_0 of a HEDGING STRATEGY Φ:

$$G(S) = V_T(\Phi, v_0; S).$$

- A strategy Φ is ARBITRAGE (free lunch) if

$$\mathbf{P}\left[V_T(\Phi, 0; S) \geq 0\right] = 1 \quad \text{and} \quad \mathbf{P}\left[V_T(\Phi, 0; S) > 0\right] > 0.$$

- If the hedging capital v_0 is not unique then there is arbitrage. Also, note that replication and arbitrage are kind of opposite notions.

3. Classical Black–Scholes pricing model

- The Stock-price process is a GEOMETRIC BROWNIAN MOTION

$$S_t = s_0 e^{\mu t + \sigma W_t - \frac{\sigma^2}{2} t}.$$

- With admissible strategies there is no arbitrage, and practically all options can be hedged.
- Let R_t be the log-return

$$R_t = \log S_t - \log S_{t-1} = \sigma \Delta W_t + \left(\mu - \frac{\sigma^2}{2}\right) \Delta t.$$

So, the log-returns are

1. independent,
2. Gaussian.

4. Stylized facts

Dictionary definition: Stylized facts are observations that have been made in so many contexts that they are widely understood to be empirical truths, to which theories must fit.

Some less-disputed stylized facts of log-returns R_t:

1. Long-range dependence: $\mathbf{Cor}[R_1, R_t] \sim t^{-\beta}$.
2. Heavy tails: $\mathbf{P}[-R_t > x] \sim x^{-\alpha_1}$, and maybe also $\mathbf{P}[R_t > x] \sim x^{-\alpha_2}$.
3. Gain/Loss asymmetry: $\mathbf{P}[-R_t > x] >> \mathbf{P}[R_t > x]$ (does not apply FX-rates, obviously).
4. Jumps.
5. Volatility clustering.

All of these stylized facts are in conflict with the Black–Scholes model, and they are ill suited for semimartingale models.

5. Robust pricing models

We introduce a class of pricing models that is invariant to the Black–Scholes model as long as option-pricing is considered. The class includes models with different stylized facts.

$(\Omega, \mathcal{F}, (S_t), (\mathcal{F}_t), \mathbf{P})$ is in the MODEL CLASS $\mathcal{M}_\sigma$ if

1. S takes values in $\mathcal{C}_{s_0,+}$,
2. the PATHWISE QUADRATIC VARIATION $\langle S \rangle$ of S is of the form
$$\mathrm{d}\langle S \rangle_t \;=\; \sigma^2 S_t^2 \,\mathrm{d}t,$$
3. for all $\varepsilon > 0$ and $\eta \in \mathcal{C}_{s_0,+}$ we have the SMALL BALL PROPERTY
$$\mathbf{P}\left[\|S - \eta\|_\infty < \varepsilon\right] > 0.$$

6. Forward integration

$\mathcal{M}_\sigma$ contains non-semimartingale models. So, we cannot use Itô integrals. However, the FORWARD INTEGRAL is economically meaningful:

- $\int_0^t \Phi_r \mathrm{d}S_r$ is the **P**-a.s. forward-sum limit

$$\lim_{n\to\infty} \sum_{\substack{t_k \in \pi_n \\ t_k \le t}} \Phi_{t_{k-1}} \left(S_{t_k} - S_{t_{k-1}} \right).$$

- Let $u \in \mathcal{C}^{1,2,1}([0,T], \mathbb{R}_+, \mathbb{R}^m)$ and $Y^1, \ldots, Y^m$ be continuous bounded variation processes. If S has quadratic variation then we have the Itô formula for $u(t, S_t, Y_t^1, \ldots, Y_t^m)$:

$$\mathrm{d}u = \frac{\partial u}{\partial t}\mathrm{d}t + \frac{\partial u}{\partial x}\mathrm{d}S + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\mathrm{d}\langle S\rangle + \sum_{i=1}^m \frac{\partial u}{\partial y_i}\mathrm{d}Y^i.$$

  This implies that the forward integral on the right hand side exists and has a continuous modification.

7. Allowed strategies

Even in the classical Black–Scholes model one restricts to 'admissible' strategies to exclude arbitrage. We shall restrict the 'admissible' strategies a little more.

A strategy Φ is ALLOWED if it is admissible and of the form

$$\Phi_t \;=\; \varphi\left(t, S_t, g_1(t,S), \ldots, g_m(t,S)\right),$$

where $\varphi \in \mathcal{C}^1([0,T] \times \mathbb{R}_+ \times \mathbb{R}^m)$ and g_k's are HINDSIGHT FACTORS:

1. $g(t,\eta) = g(t,\tilde{\eta})$ whenever $\eta(r) = \tilde{\eta}(r)$ on $r \in [0,t]$,
2. $g(\cdot,\eta)$ is of bounded variation and continuous,
3. $\left|\int_0^t f(u)\mathrm{d}g(u,\eta) - \int_0^t f(u)\mathrm{d}g(u,\tilde{\eta})\right| \le K\|f\mathbf{1}_{[0,t]}\|_\infty \|\eta - \tilde{\eta}\|_\infty$

8. A no-arbitrage and robust-hedging result

Theorem NA *There is no arbitrage with allowed strategies.*

Theorem RH *Suppose a continuous option* $G : \mathcal{C}_{s_0,+} \to \mathbb{R}$*. If* $G(\tilde{S})$ *can be hedged in one model* $\tilde{S} \in \mathcal{M}_\sigma$ *with an allowed strategy then* $G(S)$ *can be hedged in any model* $S \in \mathcal{M}_\sigma$*.*

Moreover, the hedges are – as strategies of the stock-path – independent of the model.

Moreover still, if φ *is a 'functional hedge' in one model then it is a 'functional hedge' in all models.*

Corollary PDE *In the Black–Scholes model hedges for European, Asian, and lookback-options can be constructed by using the Black–Scholes partial differential equation. These hedges hold for any model that is continuous, satisfies the small ball property, and has the same quadratic variation as the Black–Scholes model.*

9. Mixed models with stylized facts (1/2)

Consider a mixed model

$$S_t = s_0 \exp\left\{\mu t + \sigma W_t - \frac{\sigma}{2} t + \delta B_t^H - I_t^{\alpha_1} + I_t^{\alpha_2}\right\},$$

where

- B^H is a fractional Brownian motion with Hurst index $H > 0.5$. (B^H is centered Gaussian with stationary increments and variance t^{2H}.)
- I^{α_i}'s are integrated compound Poisson processes with positive heavy-tailed jumps:

  $$I_t^{\alpha_i} = \int_0^t \sum_{k:\tau_k^i \le s} U_k^i \, \mathrm{d}s,$$

  τ_k^i's are Poisson arrivals and $\mathbf{P}[U_k^i > x] \sim x^{-\alpha_i}$.
- W, B^H, I^{α_1}, and I^{α_2} are independent.

9. Mixed models with stylized facts (2/2)

Consider now stylized facts in the mixed model.

1. Long-range dependence: If I^{α_i}'s are in L^2 then

$$\mathbf{Cor}[R_1, R_t] \sim \delta^2 H(2H-1)t^{2H-2}.$$

2. Heavy tails: $\mathbf{P}[-R_t > x] \gtrsim x^{-\alpha_1}$ and $\mathbf{P}[R_t > x] \gtrsim x^{-\alpha_2}$.
3. Gain/Loss asymmetry: Obvious if $\alpha_1 < \alpha_2$.
4. Jumps: No, but can you tell the difference between jumps and heavy tails from a discrete data?
5. Volatility clustering: What is volatility? If volatility is standard deviation, we can have any kind of volatility structure: E.g. change the Poisson arrivals to clustered arrivals. If volatility (squared) is the quadratic variation then it is fixed to constant σ^2.

10. A Message: Quadratic variation and volatility

- The hedges depend only on the quadratic variation.
- The quadratic variation is a path property. It tells nothing about the probabilistic structure of the stock-price (Black and Scholes tell us the mean return is irrelevant. We boldly suggest that probability is irrelevant, as far as option-pricing is concerned).
- Don't be surprised if the implied and historical volatility do not agree: The latter is an estimate of the variance and the former is an estimate of the quadratic variation. In the Black–Scholes model these notions coincide. But that is just luck! Indeed, consider a mixed fractional Black–Scholes model $R_t = \sigma \Delta W_t + \delta \Delta B_t^H$. Then quadratic variation or R_t is σ^2, but the variance of R_t is $\sigma^2 + \delta^2$.
- Don't use the historical volatility! Instead, use either implied volatility or estimate the quadratic variation (which may be difficult).

11. Robustness beyond Black and Scholes

- Instead of taking the Black–Scholes model as reference we can consider models

$$\tilde{S}_t = s_0 \exp \tilde{X}_t,$$

where $\tilde{X}$ is continuous semimartingale with $\tilde{X}_0 = 0$.

- We can extend our robustness results to models

$$S_t = s_0 \exp X_t$$

where X is continuous $X_0 = 0$, X and $\tilde{X}$ have the same pathwise quadratic variation, and the support of $\mathbf{P} \circ X^{-1}$ is the same as the support of $\tilde{\mathbf{P}} \circ \tilde{X}^{-1}$.

- So, when option pricing is considered it does not matter whether $\tilde{S}$ or S is the model.

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