Probability is irrelevant in stochastic finance

Black-Scholes model is correct despite of stylized facts

Tommi Sottinen

Reykjavík University

30th November 2007
Outline

1 Options

2 Replication

3 Irrelevance of Probability
Outline

1 Options

2 Replication

3 Irrelevance of Probability
Let \((B_t)_{t \in [0, T]}\) be the bond, or **BANK ACCOUNT**. We work in the discounted world:

\[
B_t = 1 \quad \text{for all } t \in [0, T].
\]

The bank account is **RISKLESS**, i.e. non-random.
Let \((B_t)_{t \in [0, T]}\) be the bond, or **BANK ACCOUNT**. We work in the discounted world:

\[
B_t = 1 \quad \text{for all } t \in [0, T].
\]

The bank account is **RISKLESS**, i.e. non-random.

Let \(S = (S_t)_{t \in [0, T]}\) be the **STOCK**. The stock is **RISKY**, i.e. random.

We assume that \(S\)

1. is positive and continuous in time,
2. has non-trivial **QUADRATIC VARIATION** of form

\[
(dS_t)^2 = \sigma^2 S_t^2 \, dt.
\]
**Definition (Option)**

*Option* is simply a real-valued mapping $S \mapsto G(S)$. The asset $S$ is the *underlying* of the option $G$. 

**Example**

$G = (S_T - K)^+$ is a call-option, $G = (K - S_T)^+$ is a put-option, $G = S_T - K$ is a future. $T$ is the time of maturity and $K$ is the strike-price.
**Options**

**Definition (Option)**

**Option** is simply a real-valued mapping $S \mapsto G(S)$. The asset $S$ is the **underlying** of the option $G$.

**Example**

- $G = (S_T - K)^+$ is a **call-option**, 
- $G = (K - S_T)^+$ is a **put-option**, 
- $G = S_T - K$ is a **future**.

$T$ is the time of maturity and $K$ is the strike-price.
Outline

1. Options
2. Replication
3. Irrelevance of Probability
A trading strategy $\Phi = (\Phi_t)_{t \in [0, T]}$ is an $S$-adapted stochastic process that tells the units of the underlying asset $S$ the investor has is her portfolio at any time $t \in [0, T]$. 
A **trading strategy** \( \Phi = (\Phi_t)_{t \in [0, T]} \) is an \( S \)-adapted stochastic process that tells the units of the underlying asset \( S \) the investor has is her portfolio at any time \( t \in [0, T] \).

The **wealth** of a **self-financing** trading strategy \( \Phi \) satisfies (in the discounted world):

\[
dV_t(\Phi) = \Phi_t \, dS_t,
\]

where the differentials are of “forward type”.

---

**Replication**

**Trading strategies**
A **trading strategy** $\Phi = (\Phi_t)_{t \in [0, T]}$ is an $S$-adapted stochastic process that tells the units of the underlying asset $S$ the investor has is her portfolio at any time $t \in [0, T]$.

The **wealth** of a **self-financing** trading strategy $\Phi$ satisfies (in the discounted world):

$$dV_t(\Phi) = \Phi_t dS_t,$$

where the differentials are of “forward type”. Indeed, let $\mathcal{d}$ be the infinitesimal. Then the differential equation above can be written as

$$V_{t+\mathcal{d}} = \Phi_t S_{t+\mathcal{d}} + (V_t - \Phi_t S_t).$$
Replication principle is used to hedge and price options.

**Definition (Replication Principle)**

Let $G$ be an option. Suppose that there is a trading strategy $\Phi$ with wealth $V_t(\Phi)$ at time $t$ such that $G = V_T(\Phi)$ at time $T$. Then the price of the option $G$ at time $t$ is $V_t(\Phi)$. 

The replication requirement $G = V_T(\Phi)$ can be written as $G = V_t(\Phi) + \int_t^T \Phi_s dS_s$, where the integral is of "forward type".
Replication principle is used to hedge and price options.

**Definition (Replication Principle)**

Let $G$ be an option. Suppose that there is a trading strategy $\Phi$ with wealth $V_t(\Phi)$ at time $t$ such that $G = V_T(\Phi)$ at time $T$. Then the price of the option $G$ at time $t$ is $V_t(\Phi)$.

The replication requirement $G = V_T(\Phi)$ can be written as

$$G = V_t(\Phi) + \int_t^T \Phi_s \, dS_s,$$

where the integral is of “forward type”.
Theorem (Ito’s Formula)

The following is true when at least three terms in the equation make sense in the classical way:

$$df(t, S_t) = \frac{\partial f}{\partial t}(t, S_t)dt + \frac{\partial f}{\partial x}(t, S_t)dS_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t, S_t)(dS_t)^2.$$
Theorem (Ito’s Formula)

The following is true when at least three terms in the equation make sense in the classical way:

\[ df(t, S_t) = \frac{\partial f}{\partial t}(t, S_t)dt + \frac{\partial f}{\partial x}(t, S_t)dS_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t, S_t)(dS_t)^2. \]

Proof.

Taylor is all you need!
Let $G = g(S_T)$ be a path-independent option.
Let \( G = g(S_T) \) be a path-independent option.

Recall that

\[
(dS_t)^2 = \sigma^2 S_t^2 \, dt
\]

So, by Ito’s formula, if \( v(t, x) \) is the solution to \textbf{Black-Scholes BPDE}

\[
\frac{\partial v}{\partial t}(t, x) - \frac{\sigma^2}{2} x^2 \frac{\partial^2 v}{\partial x^2}(t, x) = 0
\]

with boundary condition \( v(T, x) = g(x) \), then
Let $G = g(S_T)$ be a path-independent option.

Recall that

$$(dS_t)^2 = \sigma^2 S_t^2 dt$$

So, by Ito's formula, if $v(t, x)$ is the solution to Black-Scholes BPDE

$$\frac{\partial v}{\partial t}(t, x) - \frac{\sigma^2}{2} x^2 \frac{\partial^2 v}{\partial x^2}(t, x) = 0$$

with boundary condition $v(T, x) = g(x)$, then

$$V_t(\Phi) = v(t, S_t)$$

and

$$\Phi_t = \frac{\partial v}{\partial x}(t, S_t).$$
Outline

1 Options

2 Replication

3 Irrelevance of Probability
Note that the prices and replications of (path-independent) options were derived from the quadratic variation property. **Probability does not come into it!**
Note that the prices and replications of (path-independent) options were derived from the quadratic variation property. **Probability does not come into it!**

It is true, by the **Feynman-Kac formula**, that

\[ v(t, x) = \mathbb{E} \left[ g \left( xe^{\sigma W_{T-t} - \frac{\sigma^2}{2} (T-t)} \right) \right], \]

where \( W \) is the standard **Brownian motion**.
Note that the prices and replications of (path-independent) options were derived from the quadratic variation property. **Probability does not come into it!**

It is true, by the **Feynman-Kac formula**, that

\[
\nu(t, x) = \mathbb{E} \left[ g \left( xe^{\sigma W_{T-t} - \frac{\sigma^2}{2} (T-t)} \right) \right],
\]

where \( W \) is the standard **Brownian motion**.

The Feynman-Kac formula can be derived e.g. from the Ito’s formula by choosing

\[
dS_t = \mu S_t \, dt + \sigma S_t \, dW_t,
\]
Note that the prices and replications of (path-independent) options were derived from the quadratic variation property. **Probability does not come into it!**

It is true, by the **Feynman-Kac formula**, that

$$v(t, x) = \mathbb{E} \left[ g \left( xe^{\sigma W_{T-t} - \frac{\sigma^2}{2} (T-t)} \right) \right],$$

where $W$ is the standard **Brownian motion**.

The Feynman-Kac formula can be derived e.g. from the Ito’s formula by choosing

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t,$$

but beware the **CLT Non Sequitur**:

*It is Gaussian. Therefore, it is a sum of small independent parts.*
The choice

\[ dS_t = \mu S_t \, dt + \sigma S_t \, dW_t, \]

corresponds to the celebrated **Black-Scholes model**.
The choice

\[ dS_t = \mu S_t \, dt + \sigma S_t \, dW_t, \]

corresponds to the celebrated **Black-Scholes model**.

The Black-Scholes model has **independent** and **normally distributed** log-returns.
The choice
\[ dS_t = \mu S_t \, dt + \sigma S_t \, dW_t, \]
corresponds to the celebrated Black-Scholes model.

The Black-Scholes model has independent and normally distributed log-returns.

In practise the log-returns are neither normally distributed nor independent. (Believe it! Everybody wants them to be, so there have been a lot of arguments why they should be. Unfortunately, the objective reality, i.e. the data, disagrees with theory.)
The choice
\[ dS_t = \mu S_t \, dt + \sigma S_t \, dW_t, \]
corresponds to the celebrated **Black-Scholes model**.

The Black-Scholes model has **independent** and **normally distributed** log-returns.

In practise the log-returns are neither normally distributed nor independent. (Believe it! Everybody wants them to be, so there have been a lot of arguments why they should be. Unfortunately, the objective reality, i.e. the data, disagrees with theory.)

But now we know that the Black-Scholes prices and replications may indeed be right even though the model is obviously wrong.
The parameter $\sigma^2$ is called the VOLATILITY of the stock.
The parameter $\sigma^2$ is called the *volatility* of the stock.

We have seen that the volatility determines the prices of the options.
The parameter $\sigma^2$ is called the **volatility** of the stock.

We have seen that the volatility determines the prices of the options.

In the Black-Scholes model the volatility is the variance of the log-returns, so one usually estimates it from the data by using the **historical volatility** i.e. one takes the empirical variance as the estimate.
The parameter $\sigma^2$ is called the **volatility** of the stock.

We have seen that the volatility determines the prices of the options.

In the Black-Scholes model the volatility is the variance of the log-returns, so one usually estimates it from the data by using the **historical volatility** i.e. one takes the empirical variance as the estimate.

**Using the historical volatility is wrong!**
The parameter $\sigma^2$ is called the **volatility** of the stock.

We have seen that the volatility determines the prices of the options.

In the Black-Scholes model the volatility is the variance of the log-returns, so one usually estimates it from the data by using the **historical volatility** i.e. one takes the empirical variance as the estimate.

**Using the historical volatility is wrong!** Indeed, consider the model

$$dS_t = S_t \sigma d(W_t + Z_t)$$

where $Z$ is a long-range-dependent fractional Brownian motion independent of the standard Brownian motion. Then the volatility of this model is $\sigma^2$, but the variance of the log-returns is $2\sigma^2$. 
- The End -