Sample path large deviations of a Gaussian process with stationary increments and regularily varying variance

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The setting

Consider a fluid queue serving at unit rate. The *input process* $Z = (Z_t : t \in \mathbb{R})$ is

- centred Gaussian with stationary increments
- $Z_0 = 0$ and the variance function σ^2 is regularly varying at infinities with index $H \in (0, 1)$, i.e. for all $t \in \mathbb{R}$

$$\lim_{\alpha \to \infty} \frac{\sigma^2(\alpha t)}{\sigma^2(\alpha)} = t^{2H}.$$

The *storage process* $V = (V_t : t \in \mathbb{R})$ is

$$V_t = \sup_{s \leq t} \left(Z_t - Z_s - (t - s) \right).$$

We are interested in the excursions of V (busy periods) and V_0 (queue length).

The large deviations of these are known in the case of fractional Brownian motion.

Fractional Brownian motion (fBm)

The fractional Brownian motion $B = B^H$ can be characterised by the following properties: it is *continuous, Gaussian, centred, of stationary increments* and *self-similar with a parameter (Hurst index)* $H \in (0, 1)$, *i.e.*

$$(B_{at} : t \in \mathbb{R}) \quad \stackrel{d}{=} \quad (a^H B_t : t \in \mathbb{R}).$$

Alternatively, one can give the covariance function

$$\mathrm{Cov}(B_t, B_s) \quad = \quad \frac{1}{2}\left(t^{2H} + s^{2H} + |t - s|^{2H}\right).$$

If $H > 1/2$ the increments of B are positively correlated, if $H < 1/2$ they are negatively correlated. The case $H = 1/2$ corresponds the standard Brownian motion.

Convergence to fBm

Set

$$Z_t^{(\alpha)} = \frac{1}{\sigma(\alpha)} Z_{\alpha t}.$$

Lemma *$Z^{(\alpha)}$ converges to fBm in finite dimensional distributions.*

Proof Obviously $\mathrm{Var} Z_t^{(\alpha)} \to t^{2H}$. Hence

$$\begin{aligned} &\mathrm{Cov}(Z_s^{(\alpha)}, Z_t^{(\alpha)}) \\ &\quad = \frac{1}{2}\left(\mathrm{Var} Z_s^{(\alpha)} + \mathrm{Var} Z_t^{(\alpha)} + \mathrm{Var} Z_{t-s}^{(\alpha)}\right) \\ &\quad \to \frac{1}{2}\left(t^{2H} + s^{2H} + |t-s|^{2H}\right). \end{aligned}$$

Since we are in the centred Gaussian case the claim follows. QED

Define $(\Omega, \|\cdot\|)$ by

$$\Omega = \left\{ \omega \in C(\mathbb{R}) : \omega_0 = 0 = \lim_{|t|\to\infty} \frac{\omega_t}{1+|t|} \right\}$$

$$\|\omega\| = \sup_{t\in\mathbb{R}} \frac{|\omega_t|}{1+|t|}.$$

Convergence to fBm, cont.

Define a majorising variance and the associated metric entropy integral

$$\begin{aligned}
\bar{\sigma}(t) &= \sup_{\alpha \geq 1} \sup_{s \leq t} \frac{\sigma^2(\alpha t)}{\sigma^2(\alpha)} \\
J(k, T) &= \int_0^k \left(\ln \left(\frac{T}{\bar{\sigma}^{-1}(\varepsilon)} + 1 \right) \right)^{\frac{1}{2}} d\varepsilon.
\end{aligned}$$

Assumptions **C**: $J(\bar{\sigma}(T), T) < \infty$ *for all* T and **B**: *there exists a sequence* $x_k \uparrow \infty$ *such that*

$$\begin{aligned}
\sum_{k=T}^{\infty} \frac{1}{1 + x_k} &< \infty \\
\sum_{k=1}^{\infty} \frac{J(\bar{\sigma}(\Delta x_k, \Delta x_k)}{1 + x_k} &< \infty
\end{aligned}$$

imply (more or less)

$$\begin{aligned}
\mathbb{P}\left(\sup_{|t-s|<\varepsilon} |Z_t^{(\alpha)} - Z_s^{(\alpha)}| > \delta \right) &\leq \exp\left(\frac{-\delta^2}{\bar{\sigma}^2(\varepsilon)} \right) \\
\mathbb{P}\left(\sup_{t \geq T} \frac{|Z_t^{(\alpha)}|}{1+t} > \varepsilon \right) &\leq \exp\left(-\varepsilon^2 T \right).
\end{aligned}$$

Convergence to fBm, cont., cont.

Lemma *$Z^{(\alpha)}$ is tight in $(\Omega, \|\cdot\|)$ iff*

$$\lim_{\delta \downarrow 0} \sup_{\alpha \geq 1} \mathbb{P}\left(\sup_{|t-s|<\delta} |Z_t^{(\alpha)} - Z_s^{(\alpha)}| > \varepsilon \right) = 0$$

$$\lim_{T \to \infty} \sup_{\alpha \geq 1} \mathbb{P}\left(\sup_{|t| \geq T} \frac{|Z_t^{(\alpha)}|}{1+|t|} > \varepsilon \right) = 0.$$

Theorem *$Z^{(\alpha)}$ converges to fBm weakly in* $(\Omega, \|\cdot\|)$.

Example The input traffic is composed of n independent streams, each of which is a fBm, with different Hurst indexes, i.e.

$$Z = \sum_{k=1}^{n} a_k B^{H_k}.$$

Counterexample ?

Large deviations

Definition A scaled family $\left(X^{(\alpha)}, v(\alpha)\right)$ satisfies the large deviations principle (LDP) in $(\Omega, \|\cdot\|)$ with rate function $I : \Omega \to [0, \infty]$ if for each closed $F \subset \Omega$ and open $G \subset \Omega$

$$\begin{aligned} \limsup_{\alpha\to\infty} \frac{1}{v(\alpha)} \ln \mathbb{P}\left(X^{(\alpha)} \in F\right) &\leq -\inf_{\omega\in F} I(\omega) \\ \liminf_{\alpha\to\infty} \frac{1}{v(\alpha)} \ln \mathbb{P}\left(X^{(\alpha)} \in G\right) &\geq -\inf_{\omega\in G} I(\omega) \end{aligned}$$

Set $v(\alpha) = \frac{\alpha^2}{\sigma^2(\alpha)}$ and consider the family

$$\left(\frac{1}{\sqrt{v(\alpha)}} Z^{(\alpha)}, v(\alpha)\right). \qquad (1)$$

Lemma *The family (1) satisfies the LDP on Ω equipped with the topology of pointwise convergece with the rate function*

$$I(x) = \sup_p \frac{1}{2} \left\langle \Gamma_p^{-1} p(x), p(x) \right\rangle \qquad (2)$$

where p is a finite dimensional projection on Ω and Γ_p is the covariance matrix of p(fBm).

Large deviations, cont.

Definition A scaled family $\left(X^{(\alpha)}, v(\alpha)\right)$ is exponentially tight in $(\Omega, \|\cdot\|)$ if for each $\ell > 0$ there exists a compact set K_ℓ such that

$$\limsup_{\alpha\to\infty} \frac{1}{v(\alpha)} \ln \mathbb{P}\left(X^{(\alpha)} \notin K_\ell\right) \quad \leq \quad -\ell.$$

Theorem *The family (1) satisfies the LDP on* $(\Omega, \|\cdot\|)$ *with the rate function (2).*

"Proof" Assumptions **C** and **B** imply the exponential tightness. The LPD can hence be lifted to the norm topology by means of the inverse contraction principle. QED

Large buffer and busy period asymptotics

Set $Q_x = \{V_0 \geq x\}$ and

$$\begin{aligned} A &= \sup\{t \leq 0 : V_t = 0\}, \\ B &= \inf\{T \geq 0 : V_t = 0\}, \\ K_T &= \{A < 0 < B\} \cap \{B - A > T\}. \end{aligned}$$

Theorem

$$\lim_{x\to\infty} \frac{\sigma^2(x)}{x^2} \ln \mathbb{P}(Z \in Q_x) = -\inf_{\omega \in Q_1} I(\omega).$$

Proof Since

$$\begin{aligned} \mathbb{P}(Z \in Q_x) &= \mathbb{P}(\sup_{t\leq 0}(Z_{xt} - xt) \geq x) \\ &= \mathbb{P}(\sup_{x\leq 0}(\frac{1}{\sqrt{v(\alpha)}} Z_t^{(\alpha)} - t) \geq 1) \\ &= \mathbb{P}(\frac{1}{\sqrt{v(\alpha)}} Z^{(\alpha)} \in Q_1) \end{aligned}$$

the claim follows from the LDP and the fact that $\inf_{\omega\in\bar{Q}_1} I(\omega) = \inf_{\omega \in Q_1^\circ} I(\omega)$. QED

Large buffer and busy period asymptotics, cont.

Theorem

$$\lim_{T\to\infty} \frac{\sigma^2(T)}{T^2} \ln \mathbb{P}(Z \in K_T) \quad = \quad -\inf_{\omega\in K_1} I(\omega).$$

Proof Since

$$\mathbb{P}(Z \in K_T) \quad = \quad \mathbb{P}(\frac{1}{\sqrt{v(\alpha)}} Z^{(\alpha)} \in K_1)$$

the claim follows from the LDP and the fact that $\inf_{\omega\in\bar{K}_1} I(\omega) = \inf_{\omega\in K_1^\circ} I(\omega)$. QED

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