What Is Volatility?

Probability is Irrelevant in Option-Pricing

Tommi Sottinen

University of Vaasa

2nd October 2008
Abstract

Volatility plays a fundamental rôle in econometric modelling and in option pricing. However, it seems that it is not clear what it is.
Volatility plays a fundamental rôle in econometric modelling and in option pricing. However, it seems that it is not clear what it is.

To illustrate the problem we construct a toy-model that incorporates long-range dependence and heavy tails to the standard Black–Scholes model while keeping the replication prices of options unchanged.
Volatility plays a fundamental rôle in econometric modelling and in option pricing. However, it seems that it is not clear what it is.

To illustrate the problem we construct a toy-model that incorporates long-range dependence and heavy tails to the standard Black–Scholes model while keeping the replication prices of options unchanged.

So, the volatility as the pricing parameter is the same as in the classical Black–Scholes model, but the historical volatility (standard deviation) is not the same as in the Black–Scholes model. Indeed, the historical volatility may not even exist.
Abstract

The moral of the story is

- The historical volatility and the implied volatility need not have anything in common.
- The probabilistic properties of the pricing model are mostly irrelevant in option-pricing.

Abstract

The moral of the story is

- The historical volatility and the implied volatility need not have anything in common.
- The probabilistic properties of the pricing model are mostly irrelevant in option-pricing.

1 Toy Model
2 Hedging with Quadratic Variation
3 Toy Model’s Volatility
4 References
1 Toy Model

2 Hedging with Quadratic Variation

3 Toy Model’s Volatility

4 References
We consider **discounted markets** with one risky asset given by the mixed model

\[ S_t = s_0 \exp \left\{ \mu t + \sigma W_t - \frac{\sigma}{2} t + \delta B_t^H - I_t^{\alpha_1} + I_t^{\alpha_2} \right\}, \]

where...
We consider discounted markets with one risky asset given by the mixed model

\[ S_t = s_0 \exp \left\{ \mu t + \sigma W_t - \frac{\sigma}{2} t + \delta B_t^H - I_t^{\alpha_1} + I_t^{\alpha_2} \right\}, \]

where

- \( B^H \) is a fractional Brownian motion with Hurst index \( H > 0.5 \).
We consider **Discounted Markets** with one risky asset given by the mixed model

\[
S_t = s_0 \exp \left\{ \mu t + \sigma W_t - \frac{\sigma}{2} t + \delta B^H_t - l^\alpha_1 + l^\alpha_2 \right\},
\]

where

- \( B^H \) is a **Fractional Brownian Motion** with Hurst index \( H > 0.5 \).
- \( l^\alpha_i \)'s are **Integrated Compound Poisson Processes** with positive heavy-tailed jumps:

\[
l^\alpha_i = \int_0^t \sum_{k: \tau^i_k \leq s} U^i_k \, ds,
\]

\( \tau^i_k \)'s are Poisson arrivals and \( P[U^i_k > x] \sim x^{-\alpha_i} \).
We consider discounted markets with one risky asset given by the mixed model

\[ S_t = s_0 \exp \left\{ \mu t + \sigma W_t - \frac{\sigma}{2} t + \delta B^H_t - l^\alpha_1 t - l^\alpha_2 t \right\} , \]

where

- \( B^H \) is a fractional Brownian motion with Hurst index \( H > 0.5 \).
- \( l^\alpha_i \)'s are integrated compound Poisson processes with positive heavy-tailed jumps:

\[ l^\alpha_i t = \int_0^t \sum_{k: \tau_{i,k} \leq s} U_{i,k}^i \, ds, \]

\( \tau_{i,k}'s \) are Poisson arrivals and \( \mathbb{P}[U_{i,k}^i > x] \sim x^{-\alpha_i} \).
- \( W, B^H, l^\alpha_1, \) and \( l^\alpha_2 \) are independent.
Stylized facts for the returns $R_t$ in the mixed model:

1. Long-range dependence: If $I_\alpha$'s are in $L_2$ then $\text{Cor}[R_1, R_t] \sim \delta^{2H - 1} t^{2H - 2}$.

2. Heavy tails: $P[-R_t > x] \gg x^{-\alpha_1}$ and $P[R_t > x] \gg x^{-\alpha_2}$.

3. Gain/Loss asymmetry: Obvious if $\alpha_1 < \alpha_2$.

4. Jumps: No, but can you tell the difference between jumps and heavy tails from a discrete data?

5. Volatility clustering: What is volatility? If volatility is the standard deviation, we can get any kind of volatility structure: Change the Poisson arrivals to clustered arrivals. If volatility (squared) is the so-called quadratic variation then it is fixed to constant $\sigma^2$. 
Stylized facts for the returns $R_t$ in the mixed model:

1. **Long-range dependence**: If $l^{\alpha_i}$'s are in $L^2$ then

   $$\text{Cor}[R_1, R_t] \sim \delta^2 H(2H - 1) t^{2H-2}.$$

2. **Heavy tails**: $P[-R_t > x] \asymp x^{-\alpha_1}$ and $P[R_t > x] \asymp x^{-\alpha_2}$.

3. **Gain/Loss asymmetry**: Obvious if $\alpha_1 < \alpha_2$.

4. **Jumps**: No, but can you tell the difference between jumps and heavy tails from a discrete data?

5. **Volatility clustering**: What is volatility? If volatility is standard deviation, we can get any kind of volatility structure: change the Poisson arrivals to clustered arrivals. If volatility (squared) is the so-called quadratic variation then it is fixed to constant $\sigma^2$. 

7 / 20
Stylized facts for the returns $R_t$ in the mixed model:

1. **Long-range dependence**: If $I^{\alpha_i}$'s are in $L^2$ then
   \[
   \text{Cor}[R_1, R_t] \sim \delta^2 H(2H - 1)t^{2H-2}.
   \]

2. **Heavy tails**: $P[-R_t > x] \gtrsim x^{-\alpha_1}$ and $P[R_t > x] \gtrsim x^{-\alpha_2}$. 

3. **Gain/Loss asymmetry**: Obvious if $\alpha_1 < \alpha_2$.

4. **Jumps**: No, but can you tell the difference between jumps and heavy tails from a discrete data?

5. **Volatility clustering**: If volatility is standard deviation, we can get any kind of volatility structure: Change the Poisson arrivals to clustered arrivals. If volatility (squared) is the so-called quadratic variation then it is fixed to constant $\sigma^2$. 


Stylized facts for the returns $R_t$ in the mixed model:

1. **Long-range dependence**: If $l^{\alpha_i}$'s are in $L^2$ then
   \[ \text{Cor}[R_1, R_t] \sim \delta^2 H(2H - 1)t^{2H-2}. \]

2. **Heavy tails**: $\mathbb{P}[-R_t > x] \gtrsim x^{-\alpha_1}$ and $\mathbb{P}[R_t > x] \gtrsim x^{-\alpha_2}$.

3. **Gain/Loss asymmetry**: Obvious if $\alpha_1 < \alpha_2$. 

Jumps: No, but can you tell the difference between jumps and heavy tails from a discrete data?

Volatility clustering: What is volatility? If volatility is standard deviation, we can get any kind of volatility structure: Change the Poisson arrivals to clustered arrivals. If volatility (squared) is the so-called quadratic variation then it is fixed to constant $\sigma^2$. 

Gain/Loss asymmetry: Obvious if $\alpha_1 < \alpha_2$. 

Stylized facts for the returns $R_t$ in the mixed model:

1. **Long-range dependence**: If $l^{\alpha_i}$’s are in $L^2$ then
   \[
   \text{Cor}[R_1, R_t] \sim \delta^2 H(2H - 1)t^{2H-2}.
   \]

2. **Heavy tails**: $P[-R_t > x] \gtrsim x^{-\alpha_1}$ and $P[R_t > x] \gtrsim x^{-\alpha_2}$.

3. **Gain/Loss asymmetry**: Obvious if $\alpha_1 < \alpha_2$.

4. **Jumps**: No, but can you tell the difference between jumps and heavy tails from a discrete data?
Stylized facts for the returns $R_t$ in the mixed model:

1. **Long-range dependence**: If $I^{\alpha_i}$’s are in $L^2$ then
   \[ \text{Cor}[R_1, R_t] \sim \delta^2 H(2H - 1)t^{2H-2}. \]

2. **Heavy tails**: $P[-R_t > x] \gtrsim x^{-\alpha_1}$ and $P[R_t > x] \gtrsim x^{-\alpha_2}$.

3. **Gain/Loss asymmetry**: Obvious if $\alpha_1 < \alpha_2$.

4. **Jumps**: No, but can you tell the difference between jumps and heavy tails from a discrete data?

5. **Volatility clustering**: What is volatility? If volatility is standard deviation, we can get any kind of volatility structure: Change the Poisson arrivals to clustered arrivals. If volatility (squared) is the so-called quadratic variation then it is fixed to constant $\sigma^2$. 

7 / 20
1 Toy Model

2 Hedging with Quadratic Variation

3 Toy Model’s Volatility

4 References
The forward integral is economically meaningful in the context of self-financing strategies:

Let $(\pi_n)$ is a fixed sequence of, say, dyadic partitions of $[0, T]$. Then the **Forward Integral**

$$
\int_0^t \Phi_u \, dS_u
$$

(along the sequence of partitions $(\pi_n)$) is the $\mathbb{P}$-a.s. forward-sum limit

$$
\lim_{n \to \infty} \sum_{\substack{t_k \in \pi_n \\cap \{ t_k \leq t \}}} \Phi_{t_{k-1}} \left( S_{t_k} - S_{t_{k-1}} \right)
$$

(when it exists).
Let \((\pi_n)\) is a fixed sequence of, say, dyadic partitions of \([0, T]\). Then the **Quadratic Variation**

\[
\langle S \rangle_t
\]

(along the sequence of partitions \((\pi_n)\)) is the \(P\)-a.s. limit

\[
\lim_{n \to \infty} \sum_{t_k \in \pi_n, t_k \leq t} (S_{t_k} - S_{t_{k-1}})^2
\]

(when it exists).
Some formulas for Quadratic Variation:

1. If \( \langle Y \rangle = 0 \), then \( \langle X + Y \rangle_t = \langle X \rangle_t \);

2. If \( X \) is differentiable, then \( \langle X \rangle_t = 0 \);

3. \( \langle \int_0^t f\left( X_u \right) dX_u \rangle_t = \int_0^t f\left( X_u \right)^2 d\langle X \rangle_u \);

4. \( \langle g \circ X \rangle_t = \int_0^t g'\left( X_u \right) d\langle X \rangle_u \).
Some formulas for Quadratic Variation:

1. If $\langle Y \rangle = 0$, then $\langle X + Y \rangle_t = \langle X \rangle_t$.
Hedging with Quadratic Variation

Some formulas for Quadratic Variation:

1. If $\langle Y \rangle = 0$, then $\langle X + Y \rangle_t = \langle X \rangle_t$;
2. If $X$ is differentiable, then $\langle X \rangle_t = 0$;
Some formulas for Quadratic Variation:

1. If $\langle Y \rangle = 0$, then $\langle X + Y \rangle_t = \langle X \rangle_t$;
2. If $X$ is differentiable, then $\langle X \rangle_t = 0$;
3. 
   $$\langle \int_0^t f(X_u) \, dX_u \rangle_t = \int_0^t f(X_u)^2 \, d\langle X \rangle_u$$
Some formulas for Quadratic Variation:

1. If $\langle Y \rangle = 0$, then $\langle X + Y \rangle_t = \langle X \rangle_t$;
2. If $X$ is differentiable, then $\langle X \rangle_t = 0$;
3. $\left\langle \int_0^t f(X_u) dX_u \right\rangle_t = \int_0^t f(X_u)^2 d\langle X \rangle_u$;
4. $\langle g \circ X \rangle_t = \int_0^t g'(X_u) d\langle X \rangle_u$. 
**Theorem**

Let $f \in C^{1,2}([0, T], \mathbb{R}_+)$. If $S$ has quadratic variation then we have the Itô formula

$$df(t, S_t) = f_t(t, S)dt + f_x(t, S)dS_t + \frac{1}{2} f_{xx}(t, S_t)d\langle S \rangle_t$$
**Theorem**

Let $f \in C^{1,2}([0, T], \mathbb{R}_+)$. If $S$ has quadratic variation then we have the Itô formula

$$df(t, S_t) = f_t(t, S)dt + f_x(t, S)dS_t + \frac{1}{2}f_{xx}(t, S_t)d\langle S \rangle_t$$

**Proof.**

Taylor is all you need.
**Theorem**

Let $f \in C^{1,2}([0, T], \mathbb{R}_+)$. If $S$ has quadratic variation then we have the Itô formula

$$df(t, S_t) = f_t(t, S)dt + f_x(t, S)dS_t + \frac{1}{2}f_{xx}(t, S_t)d\langle S \rangle_t$$

**Proof.**

Taylor is all you need.

**Remark**

Itô’s formula implies that the forward integral on the right hand side exists and has a continuous modification.
Theorem

Let $F(S_T)$ be a European option with maturity $T$. Let $f(t, S_t)$ satisfy the Black–Scholes BPDE

$$f_t(t, x) + \frac{\sigma^2 x^2}{2} f_{xx}(t, x) = 0, \quad f(T, x) = F(x).$$

Then $f_x(t, S_t)$ is the Delta-hedge for $F(S_T)$ and $f(0, s_0)$ is the price of the option.
**Theorem**

Let $F(S_T)$ be a European option with maturity $T$. Let $f(t, S_t)$ satisfy the **Black-Scholes BPDE**

$$f_t(t, x) + \frac{\sigma^2 x^2}{2} f_{xx}(t, x) = 0, \quad f(T, x) = F(x).$$

Then $f_x(t, S_t)$ is the Delta-hedge for $F(S_T)$ and $f(0, s_0)$ is the price of the option.

**Proof.**

Note that $d \langle S \rangle_t = \sigma^2 S_t^2 dt$, and then Itô is all you need.
We did not deal with any Equivalent Martingale Measures here. So, there are no Girsanov restrictions to the drift of $S$. 
1. We did not deal with any Equivalent Martingale Measures here. So, there are no Girsanov restrictions to the drift of $S$.

2. The Feynman-Kac connection to BPDEs tells us that

$$ f(t, x) = \mathbb{E} \left[ F(\tilde{S}_T) \mid \tilde{S}_t = x \right], $$

where $\tilde{S}$ is the Geometric Brownian Motion. This is true despite of the facts that our toy model is not log-normal, and the returns are not independent.
Outline

1 Toy Model

2 Hedging with Quadratic Variation

3 Toy Model’s Volatility

4 References
Let $R_k$’s be the log-returns:

$$R_k = \log \frac{S_k}{S_{k-1}}, \quad k = 1, 2, \ldots$$

Then

$$\text{Var}[R_k] = \sigma^2 + \delta^2 + \nu_{k1}^2 + \nu_{k2}^2,$$

where $\nu_{k1}^2, \nu_{k2}^2 \to \infty$ (possibly already $+\infty$ for finite $k$) are the variances of the increments of $I^{\alpha_1}$ and $I^{\alpha_2}$. 

Historical Volatility
Let $R_k$’s be the log-returns

$$R_k = \log \frac{S_k}{S_{k-1}}, \quad k = 1, 2, \ldots$$

Then

$$\text{Var}[R_k] = \sigma^2 + \delta^2 + v_{k1}^2 + v_{k2}^2,$$

where $v_{k1}^2, v_{k2}^2 \to \infty$ (possibly already $+\infty$ for finite $k$) are the variances of the increments of $I^{\alpha_1}$ and $I^{\alpha_2}$.

We have that

$$\frac{1}{n} \sum_{k=1}^{n} \left( R_k - \frac{1}{n} \sum_{k=1}^{n} R_k \right)^2 \to \sigma^2 + \delta^2 > \sigma^2$$

if the non-Gaussian parts vanish, and otherwise we do not have convergence at all.
Let $R_k$’s be the log-returns

$$R_k = \log \frac{S_k}{S_{k-1}}, \quad k = 1, 2, \ldots.$$ 

Then

$$\text{Var}[R_k] = \sigma^2 + \delta^2 + \nu_{k1}^2 + \nu_{k2}^2,$$

where $\nu_{k1}^2, \nu_{k2}^2 \to \infty$ (possibly already $+\infty$ for finite $k$) are the variances of the increments of $I^{\alpha_1}$ and $I^{\alpha_2}$.

We have that

$$\frac{1}{n} \sum_{k=1}^{n} \left( R_k - \frac{1}{n} \sum_{k=1}^{n} R_k \right)^2 \to \sigma^2 + \delta^2 > \sigma^2$$

if the non-Gaussian parts vanish, and otherwise we do not have convergence at all.

So, HISTORICAL VOLATILITY $\neq$ QUADRATIC VARIATION.
The options’ prices with toy-model are given by REPLICATION PRICES.
The options’ prices with toy-model are given by replication prices.

So, the implied volatility is the quadratic variation $\sigma^2$. 

Implied volatility is independent of the “smooth” parts $B_H, I_\alpha_1, I_\alpha_2$. 

The quadratic variation is independent of probabilistic properties. 

Probability is irrelevant in option pricing and replication.
The options’ prices with toy-model are given by **REPLICATION PRICES**.

So, the **IMPLIED VOLATILITY IS THE QUADRATIC VARIATION \( \sigma^2 \)**.

The **IMPLIED VOLATILITY IS INDEPENDENT OF THE “SMOOTH” PARTS \( B^H, I^{\alpha_1}, \) AND \( I^{\alpha_2} \)**.
The options’ prices with toy-model are given by replication prices.

So, the implied volatility is the quadratic variation $\sigma^2$.

The implied volatility is independent of the “smooth” parts $B^H$, $I^{\alpha_1}$, and $I^{\alpha_2}$.

The quadratic variation is independent of probabilistic properties.
The options’ prices with toy-model are given by replication prices.

So, the implied volatility is the quadratic variation $\sigma^2$.

The implied volatility is independent of the “smooth” parts $B^H$, $I^{\alpha_1}$, and $I^{\alpha_2}$.

The quadratic variation is independent of probabilistic properties.

Probability is irrelevant in option pricing and replication.
So, implied volatility is quadratic variation.
So, implied volatility is quadratic variation.

A naïve approach to estimate the implied volatility historically would then be to use

\[
\hat{\sigma}_n^2 = \frac{1}{2n} \sum_{k=1}^{2^n} \left( \frac{R_{kT}}{2^n} - \frac{1}{2n} \sum_{k=1}^{2^n} R_{kT} / 2^n \right)^2.
\]
So, implied volatility is quadratic variation.

A naïve approach to estimate the implied volatility historically would then be to use

\[ \hat{\sigma}_n^2 = \frac{1}{2n} \sum_{k=1}^{2^n} \left( R_k T/2^n - \frac{1}{2n} \sum_{k=1}^{2^n} R_k T/2^n \right)^2. \]

So, in estimating variance one lets time go to infinity, while in estimating quadratic variation one lets time-increments go to zero.
So, implied volatility is quadratic variation.

A naïve approach to estimate the implied volatility historically would then be to use

\[ \hat{\sigma}_n^2 = \frac{1}{2n} \sum_{k=1}^{2n} \left( \frac{R_k T/2^n}{n} - \frac{1}{2n} \sum_{k=1}^{2n} R_k T/2^n \right)^2. \]

So, in estimating variance one lets time go to infinity, while in estimating quadratic variation one lets time-increments go to zero.

With financial time series the naïve historical volatility estimation does not work: There is no price process in the microscopic level!
Outline

1 Toy Model

2 Hedging with Quadratic Variation

3 Toy Model’s Volatility

4 References
References


This talk: Bender, Sottinen, Valkeila (2008): Pricing by hedging and no-arbitrage beyond semimartingales.