Busy periods of a fractional Brownian type Gaussian storage

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The setting

Consider a queue fed by a zero mean Gaussian process with stationary increments and *regularly varying variance* function with index $2H$, i.e.

$$\mathbf{Var} Z_t = L(t)|t|^{2H}.$$

Here $H \in (0,1)$ and L is an even function satisfying

$$\lim_{\alpha \to \pm\infty} \frac{L(\alpha t)}{L(\alpha)} = 1$$

for all $t > 0$.

The *normalised Gaussian storage* is

$$V_t := \sup_{-\infty < s \le t} \left(Z_t - Z_s - (t-s) \right).$$

Thus V is a stationary process indicating the storage occupancy when the service rate is one.

The *busy periods* of the storage are the positive excursions of V.

The setting, cont.

Let $\mathcal{C}(\mathbb{R})$ be the space of continuous functions over $\mathbb{R}$. As the underlying probability space take

$\Omega :=$

$$\left\{\omega \in \mathcal{C}(\mathbb{R}) : \omega(0) = 0, \lim_{t \to \pm\infty} \frac{\omega(t)}{1+|t|} = 0\right\}$$

equipped with the norm

$$\|\omega\|_{\Omega} := \sup_{t \in \mathbb{R}} \frac{|\omega(t)|}{1+|t|}$$

and the corresponding Borel σ-algebra. The Probability measure $\mathbf{P}$ on Ω is such that

$$\omega(t) = Z_t(\omega).$$

(we give later assumptions on L so that $Z(\omega) \in \Omega$.)

The case of fractional Brownian motion

If $L \equiv 1$ then Z is a *fractional Brownian motion* (fBm), i.e. a centred Gaussian process with covariance function

$$R(t,s) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right).$$

Let $\mathcal{H}$ be the *Reproducing Kernel Hilbert Space* (RKHS) of Z, i.e. the space of functions $f : \mathbb{R} \to \mathbb{R}$ defined by letting

$$Z_t \mapsto R(t, \cdot)$$

span an isometry from the linear space of Z onto $\mathcal{H}$.

Remark $\mathcal{H} \subset \Omega$ as a set and the topology in $\mathcal{H}$ is finer that that of Ω.

The case of fBm, cont.

The *generalised Shilder's theorem* states:

Theorem 1 The function

$$I(\omega) = \begin{cases} \frac{1}{2}\|\omega\|^2_{\mathcal{H}}, & \text{if } \omega \in \mathcal{H}, \\ \infty, & \text{otherwise,} \end{cases}$$

is a good rate function for Z and

$$\begin{aligned} \limsup_{\alpha\to\infty} \alpha^{-1} \ln \mathbf{P}(\alpha^{-\frac{1}{2}} Z \in F) &\leq -\inf_{\omega\in F} I(\omega), \\ \liminf_{\alpha\to\infty} \alpha^{-1} \ln \mathbf{P}(\alpha^{-\frac{1}{2}} Z \in G) &\geq -\inf_{\omega\in F} I(\omega), \end{aligned}$$

for all $F \subset \Omega$ closed and $G \subset \Omega$ open, i.e. $(\alpha^{-\frac{1}{2}} Z, \alpha)_{\alpha\geq 1}$ satisfies the *Large Deviations Principle* (LDP) on Ω with rate function I.

Conditions on L

Let $\bar{\sigma}$ be a majorising variance

$$\bar{\sigma}^2(t) := \sup_{0<s<t} \sup_{\alpha \geq 1} \frac{L(\alpha s)}{L(\alpha)} s^{2H}$$

and let J be the *metric entropy integral*

$$J(\kappa, T) := \int_0^\kappa \left(\ln \left(\frac{T}{2\bar{\sigma}^{(-1)}(u)} + 1 \right) \right)^{\frac{1}{2}} \mathrm{d}u.$$

Assume

C $J(\bar{\sigma}(T), T) < \infty$ for all $T > 0$.

B there exists a sequence $(x_k)_{k \in N}$ increasing to infinity such that for all $T \in N$

$$d_T := \sum_{k=T}^{\infty} c(x_k)\bar{\sigma}(x_k) < \infty,$$

$$\sum_{k=1}^{\infty} c(x_k) J(\bar{\sigma}(\Delta x_k), \Delta x_k) < \infty,$$

where $\Delta x_k := x_{k+1} - x_k$ and $c(x) = \frac{1}{1+x}$.

Remark **C** and **B** imply $Z(\omega) \in \Omega$.

Convergence and LDP of Z

Define a family $(Z^{(\alpha)})_{\alpha \geq 1}$ by

$$Z_t^{(\alpha)} := \frac{1}{\alpha^H L(\alpha)^{\frac{1}{2}}} Z_{\alpha t}.$$

Assumptions **C** and **B** yield

Theorem 2 The processes $(Z^{(\alpha)})_{\alpha \geq 1}$ converge weakly in Ω to a fBm.

On the proof The finite dimensional convergence is obvious. Assumptions **C** and **B** are needed to prove that the family $(Z^{(\alpha)})_{\alpha \geq 1}$ is tight in Ω. $\square$

Application to busy periods, cont.

Theorem 3 The scaled family

$$\left(\frac{L(\alpha)^{\frac{1}{2}}}{\alpha^{1-H}} Z^{(\alpha)} \ , \ \frac{\alpha^{2-2H}}{L(\alpha)} \right)_{\alpha \geq 1}$$

satisfies LDP on Ω with the rate function I of a fBm.

On the proof Fix a vector $\mathbf{t} = (t_1, \ldots, t_d)$ and denote

$$\mathbf{Z}^{(\alpha)} := \left(Z_{t_1}^{(\alpha)}, \ldots, Z_{t_d}^{(\alpha)} \right).$$

Let $\Lambda^{(\alpha)}$ be the logarithm of the moment generating function of $L(\alpha)^{\frac{1}{2}} \alpha^{H-1} \mathbf{Z}^{(\alpha)}$:

$$\Lambda^{(\alpha)}(\mathbf{u}) := \ln \mathbf{E} \exp \left\langle \mathbf{u}, \frac{L(\alpha)^{\frac{1}{2}}}{\alpha^{1-H}} \mathbf{Z}^{(\alpha)} \right\rangle.$$

Application to busy periods, cont, cont.

It is easy to see that

$$\frac{\alpha^{2-2H}}{L(\alpha)}\Lambda^{(\alpha)}(\mathbf{u}) \to \frac{1}{2}\langle\Gamma\mathbf{u},\mathbf{u}\rangle\,,$$

where Γ is the covariance of

$$\mathbf{B} = \left(B^{(\alpha)}_{t_1},\ldots,B^{(\alpha)}_{t_d}\right)$$

and B is a fBm with index H. Then, for the Fenchel–Legendre tranform we have

$$\begin{aligned}\Lambda^*(\mathbf{x}) &= \sup_{\mathbf{u}\in\mathbb{R}^d}(\mathbf{ux}-\Lambda(\mathbf{u})) \\ &= \frac{1}{2}\left\langle\Gamma^{-1}\mathbf{x},\mathbf{x}\right\rangle \\ &= \frac{1}{2}\|\mathbf{x}\|^2_{\mathcal{H}}.\end{aligned}$$

The LDP in Ω equipped with projective limit topology follows now from the Gärtner–Ellis theorem.

For the full LDP on Ω we need the so-called exponential tightness which follows from assumption **C** and **B**. $\square$

Application to busy periods

Recall the storage process

$$V_t(\omega) \ := \ \sup_{-\infty < s \leq t} (\omega(t) - \omega(s) - (t - s)) \, .$$

The *busy period* containing 0 is the stochastic interval

$$[A, B] \ :=$$

$$[\sup\{t \leq 0 : V_t = 0\} \, , \inf\{t \geq 0 : V_t = 0\}] \, ,$$

if $A < 0 < B$. Otherwise the system is not busy at time 0.

Denote by

$$K_T \ := \ \{A < 0 < B, B - A > T\}$$

the set of paths for which the ongoing busy period at 0 is strictly longer than T.

Application to busy periods, cont.

Lemma For any $T \geq 1$

$$\mathbf{P}\left(Z \in K_T\right) = \mathbf{P}\left(\frac{L(T)^{\frac{1}{2}}}{T^{1-H}} Z^{(T)} \in K_1\right).$$

proof

$$\begin{aligned}
&\mathbf{P}\Big(Z \in K_T\Big) \\
&= \mathbf{P}\Big(\exists a < 0, b > (a+T)^+ \forall t \in (a,b) : \\
&\qquad Z_t - Z_a > t - a\Big) \\
&= \mathbf{P}\Big(\exists a < 0, b > (a+1)^+ \forall t \in (a,b) : \\
&\qquad Z_{Tt} - Z_{Ta} > Tt - Ta\Big) \\
&= \mathbf{P}\Big(\exists a < 0, b > (a+1)^+ \forall t \in (a,b) : \\
&\qquad T^{-1}(Z_{Tt} - Z_{Ta}) > t - a\Big) \\
&= \mathbf{P}\left(L(T)^{\frac{1}{2}} T^{H-1} Z^{(T)} \in K_1\right).
\end{aligned}$$

$\square$

Application to busy periods, cont., cont.

Theorem 4

$$\lim_{T\to\infty} \frac{L(T)}{T^{2-2H}} \ln \mathbf{P}(Z \in K_T) = -\inf_{\omega\in K_1} I(\omega),$$

where $\inf_{\omega\in K_1} I(\omega) \in [\frac{1}{2}, \frac{c_H^2}{2}]$, and

$$c_H^2 = \frac{1}{H(2H-1)(2-2H)\mathrm{B}(H-\frac{1}{2}, 2-2H)}.$$

Remark One can numerically find arbitrarily good approximations to $\inf_{\omega\in K_1} I(\omega)$ using RKHS techniques.

Example Suppose the traffic is composed of independent fBm streams with different Hurst indices, i.e.

$$Z = \sum_{k=1}^{n} a_k B^{H_k}.$$

Then assumptions **C** and **B** are satisfied and

$$\frac{L(T)}{T^{2-2H}} = \sum_{k=1}^{n} a_k^2 T^{2H_k-2}.$$

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