Conditional Full Support for Gaussian Processes with Stationary Increments

Tommi Sottinen

University of Vaasa

Kyiv, September 9, 2010
International Conference "Modern Stochastics: Theory and Applications II"

Abstract

We investigate the Conditional Full Support (CFS) property for Gaussian processes with stationary increments. Roughly speaking the law of a stochastic process has CFS if, after following the path of a process upto any stopping time, all future paths are still possible.

We give integrability conditions on the spectral measure of such a process that ensure that the process has CFS or not. In particular, the general results imply that for a process with spectral density f such that $f(\lambda) \sim c_1 \lambda^p e^{-c_2 \lambda^q}$ for $\lambda \to \infty$ (with necessarily $p < 1$ if $q = 0$), the CFS property holds if and only if $q < 1$.

This is joint work with Harry van Zanten (Eindhoven, The Netherlands) and Dario Gasbarra (Jyväskylä, Finland).

Outline

Outline

Definition (Conditional Full Support)

A continuous $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$-adapted process $X = (X_t)_{t\in[0,T]}$ has $\mathbb{F}$-Conditional Full Support ($\mathbb{F}$-CFS) if

$$\mathbf{P}\left[\sup_{t\in[\tau,T]} |X_t - \eta(t)| < \varepsilon \,\middle|\, \mathcal{F}_\tau\right] > 0$$

P-a.s. for all paths $\eta \in C_{X_\tau}[\tau, T]$, $\varepsilon > 0$, and $\mathbb{F}$-stopping times τ.

Remark

By Guasoni, Rásonyi, and Schachermayer (2008) it is enough to consider deterministic times $\tau \in [0, T]$ in the definition above.

- Guasoni, Rásonyi and Schachermayer (2008) showed that CFS implies No-Arbitrage, when proportional transaction costs are imposed.
- CFS and existence quadratic variation imply No-Arbitrage in the (frictionless) pricing framework of Bender, S. and Valkeila (2008).
- Arguably, CFS is an interesting fundamental property from a purely mathematical point of view.

Outline

Elementary Properties of CFS

Lemma (Gasbarra, S., van Zanten, Pakkanen)

1. Subfiltration *If $\mathbb{G} \subset \mathbb{F}$ and X has $\mathbb{F}$-CFS then X has $\mathbb{G}$-CFS.*
2. Equivalence *If X has $\mathbb{F}^X$-CFS and X and Y are equivalent in law then Y has $\mathbb{F}^Y$-CFS.*
3. Augmentation *X has $\mathbb{F}$-CFS if and only if X has $\bar{\mathbb{F}}$-CFS, where $\bar{\mathbb{F}}$ is the usual augmentation of $\mathbb{F}$.*
4. Independence *If Y and X are independent, and X has $\mathbb{F}^X$-CFS, then $X+Y$ has $\mathbb{F}^{X+Y}$-CFS.*

CFS for Gaussian Moving Average Processes

Lemma (Cherny)

Let X be moving average process

$$X_t = \int_{-\infty}^{t} \Big(K(t-s) - K(-s) \Big) \, \mathrm{d}W_s,$$

where W is a standard Brownian motion,

$$\int_{\mathbb{R}} \Big(K(t-s) - K(-s) \Big)^2 \mathrm{d}s < \infty,$$

K is non-vanishing. Then X has $\mathbb{F}^X$-CFS.

Outline

Recall that the covariance of a (centered) Gaussian stationary-increment process X can be written as

$$\mathbf{E}\left[X_t X_s\right] = \int_{\mathbb{R}} \frac{(e^{i\lambda t}-1)(e^{-i\lambda s}-1)}{\lambda^2}\,\mu(\mathrm{d}\lambda)$$

for some symmetric Borel measure μ called Spectral Measure. If μ admits a density f w.r.t. the Lebesgue measure, we call f Spectral Density.

If $f(\lambda) = \lambda^2|\hat{K}(\lambda)|^2$ for some $K \in L^2[0,\infty)$, then X admits a Moving Average representation:

$$X_t = \int_{-\infty}^{t} \Big(K(t-s) - K(-s)\Big)\,\mathrm{d}W_s,$$

Theorem (Gasbarra, S., van Zanten)

Let X be a stationary-increment Gaussian with spectral measure $\mu(\mathrm{d}\lambda) = \mu_s(\mathrm{d}\lambda) + f(\lambda)\mathrm{d}\lambda$.

1. *If*
$$\int_{\lambda_0}^{\infty} \frac{\log f(\lambda)}{\lambda^2}\,\mathrm{d}\lambda > -\infty,$$
then X has $\mathbb{F}^X$-CFS.
2. *If*
$$\int_{\lambda_0}^{\infty} e^{a\lambda}\,\mathrm{d}\mu(\lambda) < \infty,$$
then X does not have $\mathbb{F}^X$-CFS.

Corollary

1. *Let* $c_1, c_2 > 0$, $p \in \mathbb{R}$, $q \geq 0$,

$$f(\lambda) \sim c_1 \lambda^p e^{-c_2 \lambda^q}.$$

Then CFS holds iff $q < 1$. *In particular, taking* $q = 0$ *and* $p = 1 - 2H$, *we see that the fBm has CFS.*

2. *A continuous Brownian moving average process*

$$X_t = \int_{-\infty}^{t} \Big(K(t-s) - K(-s) \Big) \, \mathrm{d}W_s,$$

where W *is a standard Brownian motion and* $K \in L^2[0, \infty)$ *a non-trivial kernel, has CFS.*

We first prove part (1) of the main theorem and part (2) of the corollary.

By the independence lemma, we may assume that X has spectral density: $\mu(\mathrm{d}\lambda) = f(\lambda)\mathrm{d}\lambda$. We may assume that $f(\lambda) > 0$ for $|\lambda| > \lambda_0$.

Let Y have spectral density

$$g(\lambda) = \begin{cases} f(\lambda), & |\lambda| > \lambda_0, \\ \lambda^2, & |\lambda| \le \lambda_0. \end{cases}$$

Now, the law of Y is equivalent to the law of X. So, by the equivalence lemma it is enough to show that Y has CFS.

By construction,

$$\int_{\mathbb{R}} \frac{g(\lambda)}{1+\lambda^2} < \infty,$$
$$\int_{\mathbb{R}} \frac{\log g(\lambda)/\lambda^2}{1+\lambda^2}\,\mathrm{d}\lambda > -\infty.$$

Hence, $g(\lambda)/\lambda^2 = |\psi(\lambda)|^2$ for a conjugate-symmetric outer Hardy function $\psi \in \mathbb{H}^{2+}$.

So, by the Paley–Wiener theorem, $\psi = \hat{K}$ for some real $K \in L^2[0,\infty)$.

So, $g(\lambda) = \lambda^2 |\hat{K}(\lambda)|^2$, and the claims follow from the Cherny's lemma.

Let us then "prove" part (2) of the main theorem.

Now, the exponential decay of the spectral measure forces the sample paths of the process to be very smooth: even analytic.

Hence, it is ehough to observe the path on any small interval to determine the whole evolution of the path. Consequently, the process cannot have *Conditional* Full Support. □

Outline

References

- C. Bender, T. Sottinen, E. Valkeila (2008) Pricing by hedging and no-arbitrage beyond semimartingales, *Finance Stoch.* **12** (4), 441–468.
- A. Cherny (2008) Brownian moving averages have conditional full support, *Ann. Appl. Probab.* **18** (5) 1825–1830.
- D. Gasbarra, T. Sottinen, H. van Zanten (2008) Conditional full support of Gaussian processes with stationary increments, *Preprint* **487**, Department of Mathematics and Statistics, University of Helsinki.
- P. Guasoni, M. Rásonyi, W. Schachermayer (2008) Consistent price systems and face-lifting pricing under transaction costs, *Ann. Appl. Probab.* **18** (2) 491–520.
- M. Pakkanen (2010) Stochastic integrals and conditional full support. *J. Appl. Probab.*, to appear.