Generalized Gaussian Bridges of Prediction-Invertible Processes

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Abstract

A generalized bridge is the law of a stochastic process that is conditioned on multiple linear functionals of its path. We consider canonical representation of the bridges. In the canonical representation the linear spaces $\mathcal{L}_t(X) = \text{span} \{ X_s; s \leq t \}$ coinside for all $t < T$ for both the original process and its bridge representation.

A Gaussian process $X = (X_t)_{t \in [0, T]}$ is \textbf{PREDICTION-INVERTIBLE} if it can be recovered (in law, at least) from its prediction martingale:

$$X_t = \int_0^t p_T^{-1}(t, s) \, d\mathbb{E}[X_T | \mathcal{F}_s^X].$$

In discrete time all non-degenerate Gaussian processes are prediction-invertible. In continuous time this is most probably not true.
This work, S. and Yazigi (2012), combines and extends the results of Alili (2002) and Gasbarra S. and Valkeila (2007).


Outline

1. Generalized (linearly conditioned) Bridges
2. Orthogonal Representation
3. Canonical Representation for Martingales
4. Prediction-Invertible Processes
5. Canonical Representation for Prediction-Invertible Processes
6. Open Questions
1. Generalized (linearly conditioned) Bridges
2. Orthogonal Representation
3. Canonical Representation for Martingales
4. Prediction-Invertible Processes
5. Canonical Representation for Prediction-Invertible Processes
6. Open Questions
Let $X = (X_t)_{t \in [0,T]}$ be a continuous Gaussian process with positive definite covariance function $R$, mean function $m$ of bounded variation, and $X_0 = m(0)$. We consider the conditioning, or bridging, of $X$ on $N$ linear functionals $G_T = [G^i_T]_{i=1}^N$ of its paths:

$$G_T(X) = \int_0^T g(t) \, dX_t = \left[ \int_0^T g_i(t) \, dX_t \right]_{i=1}^N.$$
Let \( X = (X_t)_{t \in [0, T]} \) be a continuous Gaussian process with positive definite covariance function \( R \), mean function \( m \) of bounded variation, and \( X_0 = m(0) \). We consider the conditioning, or bridging, of \( X \) on \( N \) linear functionals \( G_T = [G_T^i]_{i=1}^N \) of its paths:

\[
G_T(X) = \int_0^T g(t) \, dX_t = \left[ \int_0^T g_i(t) \, dX_t \right]_{i=1}^N.
\]

**Remark (On Linear Independence of Conditionings)**

We assume, without any loss of generality, that the functions \( g_i \) are linearly independent. Indeed, if this is not the case then the linearly dependent, or redundant, components of \( g \) can simply be removed from the conditioning (1) below without changing it.
Generalized (linearly conditioned) Bridges

Informally, the generalized (Gaussian) bridge $X^{g:y}$ is (the law of) the (Gaussian) process $X$ conditioned on the set

$$\left\{ \int_0^T g(t) \, dX_t = y \right\} = \bigcap_{i=1}^N \left\{ \int_0^T g_i(t) \, dX_t = y_i \right\}. \quad (1)$$

The rigorous definition is given in the next slide.
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(1)

The rigorous definition is given in the next slide.

**Remark (Canonical Space Framework)**

For the sake of convenience, we will work on the **Canonical Filtered Probability Space** $(\Omega, \mathcal{F}, F, \mathbb{P})$, where $\Omega = C[0, T]$, $\mathbb{P}$ corresponds to the **Gaussian Coordinate Process** $X_t(\omega) = \omega(t)$: $\mathbb{P} = \mathbb{P}[X \in \cdot]$. The filtration $F = (\mathcal{F}_t)_{t \in [0,T]}$ is the intrinsic filtration of the coordinate process $X$ that is augmented with the null-sets and made right-continuous, and $\mathcal{F} = \mathcal{F}_T$. 
The **generalized bridge measure** $P_{g;y}$ is the regular conditional law

$$P_{g;y} = P_{g;y}[X \in \cdot] = P \left[ X \in \cdot \left| \int_0^T g(t) \, dX_t = y \right. \right].$$
The **generalized bridge measure** $\mathbb{P}^{g;y}$ is the regular conditional law

$$\mathbb{P}^{g;y} = \mathbb{P}^{g;y} \left[ X \in \cdot \right] = \mathbb{P} \left[ X \in \cdot \left| \int_0^T g(t) dX_t = y \right. \right].$$

**A representation of the generalized Gaussian bridge** is **any** process $X^{g;y}$ satisfying

$$\mathbb{P} \left[ X^{g;y} \in \cdot \right] = \mathbb{P}^{g;y} \left[ X \in \cdot \right] = \mathbb{P} \left[ X \in \cdot \left| \int_0^T g(t) dX_t = y \right. \right].$$
Remark (Technical Observations)

1. Note that the conditioning on the $\mathbb{P}$-null-set is not a problem, since the canonical space of continuous processes is small enough to admit regular conditional laws.
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2. As a measure $\mathbb{P}^g_{\cdot y}$ the generalized Gaussian bridge is unique, but it has several different representations $X^g_{\cdot y}$. Indeed, for any representation of the bridge one can combine it with any $\mathbb{P}$-measure-preserving transformation to get a new representation.
1 Generalized (linearly conditioned) Bridges
2 Orthogonal Representation
3 Canonical Representation for Martingales
4 Prediction-Invertible Processes
5 Canonical Representation for Prediction-Invertible Processes
6 Open Questions
ORTHOGONAL REPRESENTATION

Denote by $\langle \mathbf{g} \rangle$ the matrix

$$
\langle \mathbf{g} \rangle_{ij} := \langle g_i, g_j \rangle := \text{Cov} \left[ \int_0^T g_i(t) \, dX_t, \int_0^T g_j(t) \, dX_t \right].
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$$

**Remark ($\langle \cdot \rangle$ is invertible and depend on $T$ and $R$ only)**

Note that $\langle g \rangle$ does not depend on the mean of $X$ nor on the conditioned values $y$: $\langle g \rangle$ depends only on the conditioning functions $g = [g_i]_{i=1}^N$ and the covariance $R$. 
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1. Note that $\langle \mathbf{g} \rangle$ does not depend on the mean of $X$ nor on the conditioned values $y$: $\langle \mathbf{g} \rangle$ depends only on the conditioning functions $\mathbf{g} = [g_i]_{i=1}^N$ and the covariance $R$.

2. Since $g_i$'s are linearly independent and $R$ is positive definite, the matrix $\langle \mathbf{g} \rangle$ is invertible.
The generalized Gaussian bridge $X^{g; y}$ can be represented as

$$X_t^{g; y} = X_t - \langle 1_t, g \rangle^\top \langle g \rangle^{-1} \left( \int_0^T g(u) \, dX_u - y \right).$$

(2)
Theorem (Orthogonal Representation)

The generalized Gaussian bridge $X^{g;y}$ can be represented as

$$X^g_y = X_t - \langle 1_t, g \rangle^\top \langle g \rangle^{-1} \left( \int_0^T g(u) \, dX_u - y \right). \quad (2)$$

Moreover, any generalized Gaussian bridge $X^{g:y}$ is a Gaussian process with

$$\mathbb{E} \left[ X^g_y(t) \right] = m(t) - \langle 1_t, g \rangle^\top \langle g \rangle^{-1} \left( \int_0^T g(u) \, dm(u) - y \right),$$

$$\text{Cov} \left[ X^g_y(t), X^g_y(s) \right] = \langle 1_t, 1_s \rangle - \langle 1_t, g \rangle^\top \langle g \rangle^{-1} \langle 1_s, g \rangle.$$
Corollary (Mean-Conditioning Invariance)

Let $X$ be a centered Gaussian process with $X_0 = 0$ and let $m$ be a function of bounded variation. Let $X^g := X^{g;0}$ be a bridge where the conditional functionals are conditioned to zero. Then

$$(X + m)^{g;y}_t = X^g_t + \left( m(t) - \langle 1_t, g \rangle^\top \langle g \rangle^{-1} \int_0^T g(u) \, dm(u) \right)$$

$$+ \langle 1_t, g \rangle^\top \langle g \rangle^{-1} y.$$
**Corollary (Mean-Conditioning Invariance)**

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$$(X + m)^g_{t:y} = X^g_t + \left( m(t) - \langle 1_t, g \rangle^T \langle g \rangle^{-1} \int_0^T g(u) \, dm(u) \right)$$

$$+ \langle 1_t, g \rangle^T \langle g \rangle^{-1} y.$$  

**Remark (Normalization)**

Because of the corollary above we can, and will, assume in what follows that $m = 0$ and $y = 0$. 

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**Orthogonal Representation**
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The problem with the orthogonal bridge representation (2) of $X^{g,y}$ is that in order to construct it at any point $t \in [0, T)$ one needs the whole path of the underlying process $X$ up to time $T$. In this section and the following sections we construct a bridge representation that is canonical in the following sense:
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**Definition (Canonical Representation)**

The bridge $X^{g;y}$ is of **canonical representation** if, for all $t \in [0, T)$, $X_t^{g;y} \in \mathcal{L}_t(X)$ and $X_t \in \mathcal{L}_t(X^{g;y})$. 

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**Canonical Representation for Martingales**
The problem with the orthogonal bridge representation (2) of $X_{g:y}$ is that in order to construct it at any point $t \in [0, T)$ one needs the whole path of the underlying process $X$ up to time $T$. In this section and the following sections we construct a bridge representation that is canonical in the following sense:

**Definition (Canonical Representation)**

The bridge $X_{g:y}$ is of **canonical representation** if, for all $t \in [0, T)$, $X_{g:y}^t \in \mathcal{L}_t(X)$ and $X_t \in \mathcal{L}_t(X_{g:y})$.

Here $\mathcal{L}_t(Y)$ is the closed linear subspace of $L^2(\Omega, \mathcal{F}, \mathbb{P})$ generated by the random variables $Y_s$, $s \leq t$. 
Remark (Gaussian Specialities)

Since the conditional laws of Gaussian processes are Gaussian and Gaussian spaces are linear, the assumptions $X_t^{g;y} \in \mathcal{L}_t(X)$ and $X_t \in \mathcal{L}_t(X^{g;y})$ are the same as assuming that $X_t^{g;y}$ is $\mathcal{F}_t^X$-measurable and $X_t$ is $\mathcal{F}_t^{X^{g;y}}$-measurable (and, consequently, $\mathcal{F}_t^X = \mathcal{F}_t^{X^{g;y}}$). This fact is very special to Gaussian processes. Indeed, in general conditioned processes such as generalized bridges are not linear transformations of the underlying process.
Remark (Gaussian Specialities)

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2. Another “Gaussian fact” here is that the bridges are Gaussian. In general conditioned processes do not belong to the same family of distributions as the original process.
Canonical Representation for Martingales

We shall require that the restricted measures $\mathbb{P}^{g,y}_t := \mathbb{P}^{g,y}|\mathcal{F}_t$ and $\mathbb{P}_t := \mathbb{P}|\mathcal{F}_t$ are equivalent for all $t < T$ (they are obviously singular for $t = T$). To this end we assume that the matrix

$$\langle g \rangle_{ij}(t) := \mathbb{E} \left[ (G^i_T(X) - G^i_t(X))(G^j_T(X) - G^j_t(X)) \right]$$

$$= \mathbb{E} \left[ \int_t^T g_i(s) \, dX_s \int_t^T g_j(s) \, dX_s \right]$$

is invertible for all $t < T$. 

Remark (On Notation) In the previous section we considered the matrix $\langle g \rangle$, but from now on we consider the function $\langle g \rangle(\cdot)$. Their connection is of course $\langle g \rangle = \langle g \rangle(0)$. We hope that is overloading of notation does not cause confusion to the reader.
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Canonical Representation for Martingales

Let now $M$ be a Gaussian martingale with strictly increasing BRACKET $\langle M \rangle$ and $M_0 = 0$. 

Remark (Bracket and Non-Degeneracy) Note that the bracket is strictly increasing if and only if the covariance $R$ is positive definite. Indeed, for Gaussian martingales we have $R(t, s) = \text{Var}(M_t \wedge s) = \langle M \rangle_{t \wedge s}$.

Define a Volterra kernel $\ell_g(t, s) := -g^\top(t) \langle \langle \langle g \rangle \rangle \rangle - 1(t) g(s)$.

Note that the kernel $\ell_g$ depends on the process $M$ through its covariance $\langle \langle \langle \cdot, \cdot \rangle \rangle \rangle$, and in the Gaussian martingale case we have $\langle \langle \langle g \rangle \rangle \rangle_{ij}(t) = \int_T t g_i(s) g_j(s) d\langle M \rangle_s$. 

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Define a Volterra kernel

$$\ell_g(t, s) := -g^\top(t) \langle g \rangle^{-1}(t) g(s). \quad (3)$$
Let now $M$ be a Gaussian martingale with strictly increasing \textbf{Bracket} $\langle M \rangle$ and $M_0 = 0$.

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$$\langle g \rangle_{ij}(t) = \int_t^T g_i(s)g_j(s) \, d\langle M \rangle_s.$$
The following lemma is the key observation in finding the canonical generalized bridge representation. Actually, it is a multivariate version of Proposition 6 of Gasbarra, S. and Valkeila (2007).
Canonical Representation for Martingales

The following lemma is the key observation in finding the canonical generalized bridge representation. Actually, it is a multivariate version of Proposition 6 of Gasbarra, S. and Valkeila (2007).

**Lemma (Radon-Nikodym for Bridges)**

Let $\ell_g$ be given by (3) and let $M$ be a continuous Gaussian martingale with strictly increasing bracket $\langle M \rangle$ and $M_0 = 0$. Then

$$
\log \frac{dP^g}{dP_t} = \int_0^t \int_0^s \ell_g(s, u) \, dM_u \, dM_s - \frac{1}{2} \int_0^t \left( \int_0^s \ell_g(s, u) \, dM_u \right)^2 \, d\langle M \rangle_s.
$$
**Proof.**

Let $p(\cdot; \mu, \Sigma)$ be the Gaussian density on $\mathbb{R}^N$ and let

$$
\alpha_t^g(d\mathbf{y}) := \mathbb{P} \left[ G_T(M) \in d\mathbf{y} \mid \mathcal{F}_t^{\mathcal{M}} \right].
$$
**Proof.**

Let \( p(\cdot; \mu, \Sigma) \) be the Gaussian density on \( \mathbb{R}^N \) and let

\[
\alpha_t^g(dy) := \mathbb{P} \left[ G_T(M) \in dy \mid \mathcal{F}_t^M \right].
\]

By the **Bayes’ formula** and the martingale property

\[
\frac{d\mathbb{P}_t^g}{d\mathbb{P}_t} = \frac{d\alpha_t^g(0)}{d\alpha_0^g} = \frac{p\left(0; G_t(M), \langle g \rangle(t)\right)}{p\left(0; G_0(M), \langle g \rangle(0)\right)}.
\]
Proof.

Let $p(\cdot; \mu, \Sigma)$ be the Gaussian density on $\mathbb{R}^N$ and let

$$\alpha^g_t(dy) := \mathbb{P} \left[ G_T(M) \in dy \mid \mathcal{F}_t \right].$$

By the Bayes’ formula and the martingale property

$$\frac{dP^g_t}{dP_t} = \frac{d\alpha^g_t(0)}{d\alpha^g_0(0)} = \frac{p\left(0; G_t(M), \langle g \rangle(t)\right)}{p\left(0; G_0(M), \langle g \rangle(0)\right)}.$$

Denote $F(t, M_t) := -\frac{1}{2} G_t^T \langle g \rangle^{-1}(0) G_t$ and the claim follows by using the Itô formula. \square
Corollary (Semimartingale Decomposition)

The canonical bridge representation $M^g$ satisfies the stochastic differential equation

$$dM_t = dM^g_t - \int_0^t \ell_g(t, s) dM^g_s d\langle M \rangle_t,$$

where $\ell_g$ is given by (3). Moreover $\langle M \rangle = \langle M^g \rangle$. 

Proof. The claim follows by using Girsanov's theorem.
Corollary (Semimartingale Decomposition)

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where $\ell_g$ is given by (3). Moreover $\langle M \rangle = \langle M^g \rangle$.

Proof.

The claim follows by using the Girsanov’s theorem.
Remark (Equivalence and Singularity)

Note that
\[ \int_0^{T-} \int_0^s \ell_g(s, u)^2 \, du \, ds < \infty. \]

In view of (4) this means that \( M \) and \( M^g \) are equivalent on \([0, T)\). Indeed, equation (4) can be viewed as the **Hitsuda Representation** between two equivalent Gaussian processes.
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Also note that
\[
\int_0^T \int_0^s \ell_g(s, u)^2 \, du \, ds = \infty
\]
meaning, by the Hitsuda representation theorem, that \( M \) and \( M^g \) are singular on \([0, T]\).
Next we solve the stochastic differential equation (4) of Corollary 6. In general, solving a Volterra–Stieltjes equation like (4) in a closed form is difficult.
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Of course, the general theory of Volterra equations suggests that the solution will be of the form (6) of next theorem, where $\ell^*$ is the resolvent kernel of $\ell_g$ determined by the resolvent equation (7) given below.
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Of course, the general theory of Volterra equations suggests that the solution will be of the form (6) of next theorem, where $\ell^* g$ is the resolvent kernel of $\ell g$ determined by the resolvent equation (7) given below.

Also, the general theory suggests that the resolvent kernel can be calculated implicitly by using the Neumann series. In our case the kernel $\ell g$ is a quadratic form that factorizes in its argument. This allows us to calculate the resolvent $\ell^* g$ explicitly as (5) below.
Theorem (Solution to an SDE)

Let \( s \leq t \in [0, T] \). Define the Volterra kernel

\[
\ell^*_g(t, s) := -\ell_g(t, s) \frac{\langle g \rangle'(t)}{\langle g \rangle'(s)}
\]

\[
= \frac{\langle g \rangle'(t) g^\top(t) \langle g \rangle^{-1}(t)}{\langle g \rangle'(s)} \frac{g(s)}{\langle g \rangle'(s)}.
\]

Then the bridge \( M^g \) has the canonical representation

\[
dM^g_t = dM_t - \int_0^t \ell^*_g(t, s) \, dM_s \, d\langle M \rangle_t,
\]

i.e., (6) is the solution to (4).
**Proof.**

Equation (6) is the solution to (4) if the kernel $\ell^*_g$ satisfies the **resolvent equation**

$$
\ell_g(t, s) + \ell^*_g(t, s) = \int_s^t \ell_g(t, u)\ell^*_g(u, s) \, d\langle M \rangle_u. \quad (7)
$$
Proof. 

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\ell_g(t, s) + \ell^*_g(t, s) = \int_s^t \ell_g(t, u) \ell^*_g(u, s) d\langle M\rangle_u. 
$$

(7)

Indeed, suppose (6) is the solution to (4). This means that

$$
dM_t = \left( dM_t - \int_0^t \ell^*_g(t, s) dM_s d\langle M\rangle_t \right) 
- \int_0^t \ell_g(t, s) \left( dM_s - \int_0^s \ell^*_g(s, u) dM_u d\langle M\rangle_s \right) d\langle M\rangle_t,
$$

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Proof.

In the integral form, by using the Fubini’s theorem, this means that

\[
M_t = M_t - \int_0^t \int_s^t \ell^*_g(u, s) \, d\langle M \rangle_u \, dM_s \\
- \int_0^t \int_s^t \ell_g(u, s) \, d\langle M \rangle_u \, dM_s \\
+ \int_0^t \int_s^t \int_u^s \ell_g(s, v) \ell^*_g(v, u) \, d\langle M \rangle_v \, d\langle M \rangle_u \, dM_s.
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\]

The resolvent criterion (7) follows by identifying the integrands in the \(d\langle M \rangle_u dM_s\)-integrals above.
Canonical Representation for Martingales

Proof.

In the integral form, by using the Fubini’s theorem, this means that

\[ M_t = M_t - \int_0^t \int_s^t \ell^*(u, s) \, d\langle M \rangle_u \, dM_s \]

\[ - \int_0^t \int_s^t \ell_g(u, s) \, d\langle M \rangle_u \, dM_s \]

\[ + \int_0^t \int_s^t \int_u^s \ell_{g}(s, v) \ell_{g}^*(v, u) \, d\langle M \rangle_v \, d\langle M \rangle_u \, dM_s. \]

The resolvent criterion (7) follows by identifying the integrands in the \( d\langle M \rangle_u \, dM_s \)-integrals above.

Now it is straightforward, but tedious, to check that the resolvent equation holds. We omit the details.
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Let us now consider a Gaussian process $X$ that is not a martingale. For a (Gaussian) process $X$ its **prediction martingale** is the process $\hat{X}$ defined as

$$\hat{X}_t = \mathbb{E} \left[ X_T \mid \mathcal{F}_t^X \right].$$
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Since for Gaussian processes $\hat{X}_t \in L_t(X)$, we may write, at least formally, that

$$\hat{X}_t = \int_0^t p(t, s) \, dX_s,$$

where the abstract kernel $p$ depends also on $T$ (since $\hat{X}$ depends on $T$).
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In the next definition we assume, among other things, that the kernel $p$ exists as a real, and not only formal, function. We also assume that the kernel $p$ is invertible.
**Definition (Prediction-Invertibility)**

A Gaussian process $X$ is **PREDICTION-INVERTIBLE** if there exists a kernel $p$ such that its prediction martingale $\hat{X}$ is continuous, can be represented as

$$\hat{X}_t = \int_0^t p(t, s) \, dX_s,$$

and there exists an inverse kernel $p^{-1}$ such that, for all $t \in [0, T]$, $p^{-1}(t, \cdot) \in L^2([0, T], d\langle \hat{X} \rangle)$ and $X$ can be recovered from $\hat{X}$ by

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In general it seems to be a difficult problem to determine whether a Gaussian process is prediction-invertible or not. In the discrete time non-degenerate case all Gaussian processes are prediction-invertible. In continuous time the situation is more difficult, as the next example illustrates.
Remark

In general it seems to be a difficult problem to determine whether a Gaussian process is prediction-invertible or not. In the discrete time non-degenerate case all Gaussian processes are prediction-invertible. In continuous time the situation is more difficult, as the next example illustrates.

Nevertheless, we can immediately see that we must have

\[ R(t, s) = \int_0^{t \wedge s} p^{-1}(t, u) p^{-1}(s, u) \, d\langle \hat{X} \rangle_u, \]

where

\[ \langle \hat{X} \rangle_u = \text{Var} \left( \mathbb{E} [X_T | \mathcal{F}_u] \right). \]

However, this criterion does not seem to be very helpful in practice.
Consider the Gaussian slope $X_t = t\xi$, $t \in [0, T]$, where $\xi$ is a standard normal random variable. Now, if we consider the “raw filtration” $\mathcal{G}_t^X = \sigma(X_s; s \leq t)$, then $X$ is not prediction invertible. Indeed, then $\hat{X}_0 = 0$ but $\hat{X}_t = X_T$, if $t \in (0, T]$. So, $\hat{X}$ is not continuous. On the other hand, the augmented filtration is simply $\mathcal{F}_t^X = \sigma(\xi)$ for all $t \in [0, T]$. So, $\hat{X} = X_T$. Note, however, that in both cases the slope $X$ can be recovered from the prediction martingale: $X_t = \frac{t}{T} \hat{X}_t$. 
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**Definition (Linear Extensions of Kernels)**

Let $X$ be prediction-invertible. Let $P$ and $P^{-1}$ extend the relations $P[1_t] = p(t, \cdot)$ and $P^{-1}[1_t] = p^{-1}(t, \cdot)$ linearly.
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**Lemma (Abstract vs. Concrete Wiener Integrals)**

For $f$ and $\hat{g}$ nice enough

$$\int_0^T f(t) \, dX_t = \int_0^T P^{-1}[f](t) \, d\hat{X}_t,$$

$$\int_0^T \hat{g}(t) \, d\hat{X}_t = \int_0^T P[\hat{g}](t) \, dX_t.$$
Outline

1. Generalized (linearly conditioned) Bridges
2. Orthogonal Representation
3. Canonical Representation for Martingales
4. Prediction-Invertible Processes
5. Canonical Representation for Prediction-Invertible Processes
6. Open Questions
The next (our main) theorem follows basically by rewriting

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**Theorem (Canonical Representation)**

Let \( X \) be prediction-invertible Gaussian process. Then
\[ X_t^g = X_t - \int_0^t \int_s^t P^{-1}(t, u) P \left[ \hat{\ell}^* (u, \cdot) \right] (s) \langle \hat{X} \rangle_u \, dX_s, \]
where \( \hat{g} = P^{-1}[g] \) and
\[ \hat{\ell}^*_g (u, v) = \langle \hat{g} \rangle^\hat{X} |(u) \hat{g}^\top (u)(\hat{g})^\top)^{-1} (u) \, \frac{\hat{g}(v)}{|\langle \hat{g} \rangle^\hat{X}|(v)}. \]
Proof.

In the prediction-martingale level we have

\[ d\hat{X}_{g} = d\hat{X}_{s} - \int_{0}^{s} \hat{\ell}^{*}(s, u) d\hat{X}_{u} d\langle \hat{X} \rangle_{s}. \]
**Proof.**

In the prediction-martingale level we have

\[ d\hat{X}_{\hat{g}} = d\hat{X}_{s} - \int_{0}^{s} \hat{\ell}^{*}(s, u) d\hat{X}_{u} d\langle \hat{X} \rangle_{s}. \]

Now, by operating with \( p^{-1} \) and \( P \) we get

\[
X_{t}^{g} = X_{t} - \int_{0}^{t} p^{-1}(t, s) \left( \int_{0}^{s} \hat{\ell}^{*}(s, u) d\hat{X}_{u} \right) d\langle \hat{X} \rangle_{s} \\
= X_{t} - \int_{0}^{t} p^{-1}(t, s) \int_{0}^{s} P \left[ \hat{\ell}^{*}(s, \cdot) \right] (u) dX_{u} d\langle \hat{X} \rangle_{s}.
\]

Finally, the claim follows by using the Fubini’s theorem.
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– The End –

Thank you for your attention!