Generalized Gaussian Bridges of Prediction-Invertible Processes

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Modern Stochastics: Theory and Applications III

September 10, 2012, Kyiv, Ukraine

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Abstract

A generalized bridge is the law of a stochastic process that is conditioned on multiple linear functionals of its path. We consider canonical representation of the bridges. In the canonical representation the linear spaces $\mathcal{L}_t(X) = \overline{\mathrm{span}}\,\{X_s; s \leq t\}$ coinside for all $t < T$ for both the original process and its bridge representation.

A Gaussian process $X = (X_t)_{t\in[0,T]}$ is PREDICTION-INVERTIBLE if it can be recovered (in law, at least) from its prediction martingale:

$$X_t = \int_0^t p_T^{-1}(t,s)\,\mathrm{d}\mathbb{E}[X_T|\mathcal{F}_s^X].$$

In discrete time all non-degenerate Gaussian processes are prediction-invertible. In continuous time this is most probably not true.

References

This work, S. and Yazigi (2012), combines and extends the results of Alili (2002) and Gasbarra S. and Valkeila (2007).

- Alili. *Canonical decompositions of certain generalized Brownian bridges.* Electron. Comm. Probab., 7, 2002, 27–36.
- Gasbarra, S. and Valkeila. *Gaussian bridges.* Stochastic analysis and applications, Abel Symp. 2, 2007, 361–382.
- S. and Yazigi *Generalized Gaussian Bridges.* Preprint `arXiv:1205.3405v1`, May 2012.

Outline

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Generalized (linearly conditioned) Bridges

Let $X = (X_t)_{t\in[0,T]}$ be a continuous Gaussian process with positive definite covariance function R, mean function m of bounded variation, and $X_0 = m(0)$. We consider the conditioning, or bridging, of X on N linear functionals $\mathbf{G}_T = [G_T^i]_{i=1}^N$ of its paths:

$$\mathbf{G}_T(X) = \int_0^T \mathbf{g}(t)\,\mathrm{d}X_t = \left[\int_0^T g_i(t)\,\mathrm{d}X_t\right]_{i=1}^N .$$

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Remark (On Linear Independence of Conditionings)

We assume, without any loss of generality, that the functions g_i are linearly independent. Indeed, if this is not the case then the linearly dependent, or redundant, components of $\mathbf{g}$ can simply be removed from the conditioning (1) below without changing it.

Generalized (linearly conditioned) Bridges

Informally, the generalized (Gaussian) bridge $X^{\mathbf{g};\mathbf{y}}$ is (the law of) the (Gaussian) process X conditioned on the set

$$\left\{ \int_0^T \mathbf{g}(t)\,\mathrm{d}X_t = \mathbf{y} \right\} = \bigcap_{i=1}^{N} \left\{ \int_0^T g_i(t)\,\mathrm{d}X_t = y_i \right\}. \qquad (1)$$

The rigorous definition is given in the next slide.

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The rigorous definition is given in the next slide.

Remark (Canonical Space Framework)

For the sake of convenience, we will work on the canonical filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\Omega = C[0,T]$, $\mathbb{P}$ corresponds to the Gaussian coordinate process $X_t(\omega) = \omega(t)$: $\mathbb{P} = \mathbb{P}[X \in \cdot\,]$. The filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ is the intrinsic filtration of the coordinate process X that is augmented with the null-sets and made right-continuous, and $\mathcal{F} = \mathcal{F}_T$.

Generalized (linearly conditioned) Bridges

Definition (Generalized Gaussian Bridge)

The generalized bridge measure $\mathbb{P}^{\mathbf{g};\mathbf{y}}$ is the regular conditional law

$$\mathbb{P}^{\mathbf{g};\mathbf{y}} = \mathbb{P}^{\mathbf{g};\mathbf{y}}\left[X \in \cdot\,\right] = \mathbb{P}\left[X \in \cdot \,\middle|\, \int_0^T \mathbf{g}(t)\,\mathrm{d}X_t = \mathbf{y}\right].$$

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A representation of the generalized Gaussian bridge is any process $X^{\mathbf{g};\mathbf{y}}$ satisfying

$$\mathbb{P}\left[X^{\mathbf{g};\mathbf{y}} \in \cdot\,\right] = \mathbb{P}^{\mathbf{g};\mathbf{y}}\left[X \in \cdot\,\right] = \mathbb{P}\left[X \in \cdot \,\middle|\, \int_0^T \mathbf{g}(t)\,\mathrm{d}X_t = \mathbf{y}\right].$$

Generalized (linearly conditioned) Bridges

Remark (Technical Observations)

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2. As a MEASURE $\mathbb{P}^{\mathbf{g};\mathbf{y}}$ the generalized Gaussian bridge is unique, but it has several different REPRESENTATIONS $X^{\mathbf{g};\mathbf{y}}$. Indeed, for any representation of the bridge one can combine it with any $\mathbb{P}$-MEASURE-PRESERVING transformation to get a new representation.

Outline

Orthogonal Representation

Denote by $\langle\!\langle \mathbf{g} \rangle\!\rangle$ the matrix

$$\langle\!\langle \mathbf{g} \rangle\!\rangle_{ij} := \langle\!\langle g_i, g_j \rangle\!\rangle := \mathbb{C}\mathrm{ov}\left[\int_0^T g_i(t)\,\mathrm{d}X_t\,,\ \int_0^T g_j(t)\,\mathrm{d}X_t\right].$$

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Remark ($\langle\!\langle \cdot \rangle\!\rangle$ is invertible and depend on T and R only)

1. Note that $\langle\!\langle \mathbf{g} \rangle\!\rangle$ does not depend on the mean of X nor on the conditioned values $\mathbf{y}$: $\langle\!\langle \mathbf{g} \rangle\!\rangle$ depends only on the conditioning functions $\mathbf{g} = [g_i]_{i=1}^N$ and the covariance R.

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2. Since g_i's are linearly independent and R is positive definite, the matrix $\langle\!\langle \mathbf{g} \rangle\!\rangle$ is invertible.

Orthogonal Representation

Theorem (Orthogonal Representation)

The generalized Gaussian bridge $X^{\mathbf{g};\mathbf{y}}$ can be represented as

$$X_t^{\mathbf{g};\mathbf{y}} = X_t - \langle\!\langle 1_t, \mathbf{g} \rangle\!\rangle^\top \langle\!\langle \mathbf{g} \rangle\!\rangle^{-1} \left(\int_0^T \mathbf{g}(u)\,\mathrm{d}X_u - \mathbf{y} \right). \qquad (2)$$

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Moreover, any generalized Gaussian bridge $X^{\mathbf{g};\mathbf{y}}$ is a Gaussian process with

$$\begin{aligned} \mathbb{E}\left[X_t^{\mathbf{g};\mathbf{y}}\right] &= m(t) - \\ &\quad \langle\!\langle 1_t, \mathbf{g} \rangle\!\rangle^\top \langle\!\langle \mathbf{g} \rangle\!\rangle^{-1} \left(\int_0^T \mathbf{g}(u)\,\mathrm{d}m(u) - \mathbf{y} \right), \\ \mathbb{C}\mathrm{ov}\left[X_t^{\mathbf{g};\mathbf{y}}, X_s^{\mathbf{g};\mathbf{y}}\right] &= \langle\!\langle 1_t, 1_s \rangle\!\rangle - \langle\!\langle 1_t, \mathbf{g} \rangle\!\rangle^\top \langle\!\langle \mathbf{g} \rangle\!\rangle^{-1} \langle\!\langle 1_s, \mathbf{g} \rangle\!\rangle. \end{aligned}$$

Orthogonal Representation

Corollary (Mean-Conditioning Invariance)

Let X be a centered Gaussian process with $X_0 = 0$ and let m be a function of bounded variation. Let $X^{\mathbf{g}} := X^{\mathbf{g};\mathbf{0}}$ be a bridge where the conditional functionals are conditioned to zero. Then

$$\begin{aligned}(X+m)_t^{\mathbf{g};\mathbf{y}} &= X_t^{\mathbf{g}} + \left(m(t) - \langle\!\langle 1_t, \mathbf{g} \rangle\!\rangle^\top \langle\!\langle \mathbf{g} \rangle\!\rangle^{-1} \int_0^T \mathbf{g}(u)\,\mathrm{d}m(u) \right) \\ &\quad + \langle\!\langle 1_t, \mathbf{g} \rangle\!\rangle^\top \langle\!\langle \mathbf{g} \rangle\!\rangle^{-1} \mathbf{y}.\end{aligned}$$

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\end{aligned}
$$

Remark (Normalization)

Because of the corollary above we can, and will, assume in what follows that $m = 0$ and $\mathbf{y} = \mathbf{0}$.

Outline

Canonical Representation for Martingales

The problem with the orthogonal bridge representation (2) of $X^{\mathbf{g};\mathbf{y}}$ is that in order to construct it at any point $t \in [0, T)$ one needs the whole path of the underlying process X up to time T. In this section and the following sections we construct a bridge representation that is canonical in the following sense:

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Definition (Canonical Representation)

The bridge $X^{\mathbf{g};\mathbf{y}}$ is of canonical representation if, for all $t \in [0, T)$, $X_t^{\mathbf{g};\mathbf{y}} \in \mathcal{L}_t(X)$ and $X_t \in \mathcal{L}_t(X^{\mathbf{g};\mathbf{y}})$.

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Here $\mathcal{L}_t(Y)$ is the closed linear subspace of $L^2(\Omega, \mathcal{F}, \mathbb{P})$ generated by the random variables Y_s, $s \leq t$.

Remark (Gaussian Specialities)

1. Since the conditional laws of Gaussian processes are Gaussian and Gaussian spaces are linear, the assumptions $X_t^{\mathbf{g};\mathbf{y}} \in \mathcal{L}_t(X)$ and $X_t \in \mathcal{L}_t(X^{\mathbf{g};\mathbf{y}})$ are the same as assuming that $X_t^{\mathbf{g};\mathbf{y}}$ is $\mathcal{F}_t^X$-measurable and X_t is $\mathcal{F}_t^{X^{\mathbf{g};\mathbf{y}}}$-measurable (and, consequently, $\mathcal{F}_t^X = \mathcal{F}_t^{X^{\mathbf{g};\mathbf{y}}}$). This fact is very special to Gaussian processes. Indeed, in general conditioned processes such as generalized bridges are not linear transformations of the underlying process.

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2. Another "Gaussian fact" here is that the bridges are Gaussian. In general conditioned processes do not belong to the same family of distributions as the original process.

Canonical Representation for Martingales

We shall require that the restricted measures $\mathbb{P}_t^{\mathbf{g},\mathbf{y}} := \mathbb{P}^{\mathbf{g};\mathbf{y}}|\mathcal{F}_t$ and $\mathbb{P}_t := \mathbb{P}|\mathcal{F}_t$ are equivalent for all $t < T$ (they are obviously singular for $t = T$). To this end we assume that the matrix

$$\begin{aligned}
\langle\!\langle \mathbf{g} \rangle\!\rangle_{ij}(t) &:= \mathbb{E}\left[\Big(G_T^i(X) - G_t^i(X)\Big)\Big(G_T^j(X) - G_t^j(X)\Big)\right] \\
&= \mathbb{E}\left[\int_t^T g_i(s)\,\mathrm{d}X_s \int_t^T g_j(s)\,\mathrm{d}X_s\right]
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Remark (On Notation)

In the previous section we considered the matrix $\langle\!\langle \mathbf{g} \rangle\!\rangle$, but from now on we consider the function $\langle\!\langle \mathbf{g} \rangle\!\rangle(\cdot)$. Their connection is of course $\langle\!\langle \mathbf{g} \rangle\!\rangle = \langle\!\langle \mathbf{g} \rangle\!\rangle(0)$. We hope that is overloading of notation does not cause confusion to the reader.

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Remark (Bracket and Non-Degeneracy)

Note that the bracket is strictly increasing if and only if the covariance R is positive definite. Indeed, for Gaussian martingales we have $R(t, s) = \mathbb{V}\mathrm{ar}(M_{t \wedge s}) = \langle M \rangle_{t \wedge s}$.

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Define a Volterra kernel

$$\ell_{\mathbf{g}}(t,s) := -\mathbf{g}^\top(t)\,\langle\!\langle \mathbf{g} \rangle\!\rangle^{-1}(t)\,\mathbf{g}(s). \tag{3}$$

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Define a Volterra kernel

$$\ell_{\mathbf{g}}(t,s) := -\mathbf{g}^{\top}(t)\,⟪\mathbf{g}⟫^{-1}(t)\,\mathbf{g}(s). \tag{3}$$

Note that the kernel $\ell_{\mathbf{g}}$ depends on the process M through its covariance $⟪\cdot,\cdot⟫$, and in the Gaussian martingale case we have

$$⟪\mathbf{g}⟫_{ij}(t) = \int_t^T g_i(s) g_j(s)\,\mathrm{d}\langle M \rangle_s.$$

Canonical Representation for Martingales

The following lemma is the key observation in finding the canonical generalized bridge representation. Actually, it is a multivariate version of Proposition 6 of Gasbarra, S. and Valkeila (2007).

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Lemma (Radon-Nikodym for Bridges)

Let $\ell_{\mathbf{g}}$ be given by (3) and let M be a continuous Gaussian martingale with strictly increasing bracket $\langle M\rangle$ and $M_0 = 0$. Then

$$\log \frac{\mathrm{d}\mathbb{P}^{\mathbf{g}}_t}{\mathrm{d}\mathbb{P}_t} = \int_0^t \int_0^s \ell_{\mathbf{g}}(s,u)\,\mathrm{d}M_u \mathrm{d}M_s - \frac{1}{2}\int_0^t \left(\int_0^s \ell_{\mathbf{g}}(s,u)\,\mathrm{d}M_u\right)^2 \mathrm{d}\langle M\rangle_s.$$

Canonical Representation for Martingales

Proof.

Let $p(\cdot;\boldsymbol{\mu},\boldsymbol{\Sigma})$ be the Gaussian density on $\mathbb{R}^N$ and let

$$\alpha_t^{\mathbf{g}}(\mathrm{d}\mathbf{y}) := \mathbb{P}\left[\mathbf{G}_T(M) \in \mathrm{d}\mathbf{y} \;\middle|\; \mathcal{F}_t^M\right].$$

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By the Bayes' formula and the martingale property

$$\frac{\mathrm{d}\mathbb{P}_t^{\mathbf{g}}}{\mathrm{d}\mathbb{P}_t} = \frac{\mathrm{d}\alpha_t^{\mathbf{g}}}{\mathrm{d}\alpha_0^{\mathbf{g}}}(\mathbf{0}) = \frac{p\Big(\mathbf{0};\mathbf{G}_t(M),\langle\!\langle\mathbf{g}\rangle\!\rangle(t)\Big)}{p\Big(\mathbf{0};\mathbf{G}_0(M),\langle\!\langle\mathbf{g}\rangle\!\rangle(0)\Big)}.$$

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Denote $F(t, M_t) := -\frac{1}{2}\mathbf{G}_t^\top \langle\!\langle \mathbf{g} \rangle\!\rangle^{-1}(0)\mathbf{G}_t$ and the claim follows by using the Itô formula. $\square$

Canonical Representation for Martingales

Corollary (Semimartingale Decomposition)

The canonical bridge representation $M^{\mathbf{g}}$ satisfies the stochastic differential equation

$$\mathrm{d}M_t = \mathrm{d}M_t^{\mathbf{g}} - \int_0^t \ell_{\mathbf{g}}(t,s)\,\mathrm{d}M_s^{\mathbf{g}}\,\mathrm{d}\langle M\rangle_t, \tag{4}$$

where $\ell_{\mathbf{g}}$ is given by (3). Moreover $\langle M\rangle = \langle M^{\mathbf{g}}\rangle$.

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Proof.

The claim follows by using the Girsanov's theorem. □

Remark (Equivalence and Singularity)

Note that

$$\int_0^{T-}\int_0^s \ell_{\mathbf{g}}(s,u)^2\,\mathrm{d}u\,\mathrm{d}s < \infty.$$

In view of (4) this means that M and $M^{\mathbf{g}}$ are equivalent on $[0,T)$. Indeed, equation (4) can be viewed as the Hitsuda representation between two equivalent Gaussian processes.

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Also note that

$$\int_0^{T}\int_0^s \ell_{\mathbf{g}}(s,u)^2\,\mathrm{d}u\,\mathrm{d}s = \infty$$

meaning, by the Hitsuda representation theorem, that M and $M^{\mathbf{g}}$ are singular on $[0,T]$.

Next we solve the stochastic differential equation (4) of Corollary 6. In general, solving a VOLTERRA–STIELTJES equation like (4) in a closed form is difficult.

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Of course, the general theory of Volterra equations suggests that the solution will be of the form (6) of next theorem, where $\ell^*_{\mathbf{g}}$ is the resolvent kernel of $\ell_{\mathbf{g}}$ determined by the resolvent equation (7) given below.

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Of course, the general theory of Volterra equations suggests that the solution will be of the form (6) of next theorem, where $\ell^*_{\mathbf{g}}$ is the resolvent kernel of $\ell_{\mathbf{g}}$ determined by the resolvent equation (7) given below.

Also, the general theory suggests that the resolvent kernel can be calculated implicitly by using the Neumann series. In our case the kernel $\ell_{\mathbf{g}}$ is a quadratic form that factorizes in its argument. This allows us to calculate the resolvent $\ell^*_{\mathbf{g}}$ explicitly as (5) below.

Canonical Representation for Martingales

Theorem (Solution to an SDE)

Let $s \le t \in [0,T]$. Define the Volterra kernel

$$\begin{aligned} \ell^*_{\mathbf{g}}(t,s) &:= -\ell_{\mathbf{g}}(t,s)\frac{|\langle\!\langle \mathbf{g}\rangle\!\rangle|(t)}{|\langle\!\langle \mathbf{g}\rangle\!\rangle|(s)} \\ &= |\langle\!\langle \mathbf{g}\rangle\!\rangle|(t)\mathbf{g}^\top(t)\langle\!\langle \mathbf{g}\rangle\!\rangle^{-1}(t)\,\frac{\mathbf{g}(s)}{|\langle\!\langle \mathbf{g}\rangle\!\rangle|(s)}. \end{aligned} \tag{5}$$

Then the bridge $M^{\mathbf{g}}$ has the canonical representation

$$\mathrm{d}M^{\mathbf{g}}_t = \mathrm{d}M_t - \int_0^t \ell^*_{\mathbf{g}}(t,s)\,\mathrm{d}M_s\,\mathrm{d}\langle M\rangle_t, \tag{6}$$

i.e., (6) is the solution to (4).

Canonical Representation for Martingales

Proof.

Equation (6) is the solution to (4) if the kernel $\ell^*_{\mathbf{g}}$ satisfies the RESOLVENT EQUATION

$$\ell_{\mathbf{g}}(t,s) + \ell^*_{\mathbf{g}}(t,s) = \int_s^t \ell_{\mathbf{g}}(t,u)\ell^*_{\mathbf{g}}(u,s)\,\mathrm{d}\langle M\rangle_u. \tag{7}$$

Canonical Representation for Martingales

Proof.

Equation (6) is the solution to (4) if the kernel $\ell^*_{\mathbf{g}}$ satisfies the resolvent equation

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Indeed, suppose (6) is the solution to (4). This means that

$$\begin{aligned}\mathrm{d}M_t &= \left(\mathrm{d}M_t - \int_0^t \ell^*_{\mathbf{g}}(t,s)\,\mathrm{d}M_s\,\mathrm{d}\langle M\rangle_t\right) \\ &\quad - \int_0^t \ell_{\mathbf{g}}(t,s)\left(\mathrm{d}M_s - \int_0^s \ell^*_{\mathbf{g}}(s,u)\,\mathrm{d}M_u\,\mathrm{d}\langle M\rangle_s\right)\mathrm{d}\langle M\rangle_t,\end{aligned}$$

□

Canonical Representation for Martingales

Proof.

In the integral form, by using the Fubini's theorem, this means that

$$\begin{aligned} M_t &= M_t - \int_0^t \int_s^t \ell^*_{\mathbf{g}}(u,s)\,\mathrm{d}\langle M\rangle_u \mathrm{d}M_s \\ &\quad - \int_0^t \int_s^t \ell_{\mathbf{g}}(u,s)\,\mathrm{d}\langle M\rangle_u \mathrm{d}M_s \\ &\quad + \int_0^t \int_s^t \int_u^s \ell_{\mathbf{g}}(s,v)\ell^*_{\mathbf{g}}(v,u)\mathrm{d}\langle M\rangle_v\,\mathrm{d}\langle M\rangle_u \mathrm{d}M_s. \end{aligned}$$

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The resolvent criterion (7) follows by identifying the integrands in the $\mathrm{d}\langle M\rangle_u \mathrm{d}M_s$-integrals above.

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The resolvent criterion (7) follows by identifying the integrands in the $\mathrm{d}\langle M\rangle_u \mathrm{d}M_s$-integrals above.

Now it is straightforward, but tedious, to check that the resolvent equation holds. We omit the details. □

Outline

Prediction-Invertible Processes

Let us now consider a Gaussian process X that is not a martingale. For a (Gaussian) process X its PREDICTION MARTINGALE is the process $\hat{X}$ defined as

$$\hat{X}_t = \mathbb{E}\left[X_T \middle| \mathcal{F}_t^X\right].$$

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Since for Gaussian processes $\hat{X}_t \in \mathcal{L}_t(X)$, we may write, at least formally, that

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In the next definition we assume, among other things, that the kernel p exists as a real, and not only formal, function. We also assume that the kernel p is invertible.

Prediction-Invertible Processes

Definition (Prediction-Invertibility)

A Gaussian process X is prediction-invertible if there exists a kernel p such that its prediction martingale $\hat{X}$ is continuous, can be represented as

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and there exists an inverse kernel p^{-1} such that, for all $t \in [0, T]$, $p^{-1}(t,\cdot) \in L^2([0,T], \mathrm{d}\langle \hat{X} \rangle)$ and X can be recovered from $\hat{X}$ by

$$X_t = \int_0^t p^{-1}(t,s)\,\mathrm{d}\hat{X}_s.$$

Prediction-Invertible Processes

Remark

In general it seems to be a difficult problem to determine whether a Gaussian process is prediction-invertible or not. In the discrete time non-degenerate case all Gaussian processes are prediction-invertible. In continuous time the situation is more difficult, as the next example illustrates.

Prediction-Invertible Processes

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In general it seems to be a difficult problem to determine whether a Gaussian process is prediction-invertible or not. In the discrete time non-degenerate case all Gaussian processes are prediction-invertible. In continuous time the situation is more difficult, as the next example illustrates.

Nevertheless, we can immediately see that we must have

$$R(t,s) = \int_0^{t\wedge s} p^{-1}(t,u)\, p^{-1}(s,u)\, \mathrm{d}\langle \hat{X}\rangle_u,$$

where

$$\langle \hat{X}\rangle_u = \mathbb{V}\mathrm{ar}\left(\mathbb{E}\left[X_T | \mathcal{F}_u\right]\right).$$

However, this criterion does not seem to be very helpful in practice.

Prediction-Invertible Processes

Example (Fine Points of Prediction-Invertibility)

Consider the Gaussian slope $X_t = t\xi$, $t \in [0, T]$, where ξ is a standard normal random variable. Now, if we consider the "raw filtration" $\mathcal{G}_t^X = \sigma(X_s; s \leq t)$, then X is not prediction invertible. Indeed, then $\hat{X}_0 = 0$ but $\hat{X}_t = X_T$, if $t \in (0, T]$. So, $\hat{X}$ is not continuous. On the other hand, the augmented filtration is simply $\mathcal{F}_t^X = \sigma(\xi)$ for all $t \in [0, T]$. So, $\hat{X} = X_T$. Note, however, that in both cases the slope X can be recovered from the prediction martingale: $X_t = \frac{t}{T}\hat{X}_t$.

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Lemma (Abstract vs. Concrete Wiener Integrals)

For f and $\hat{g}$ NICE ENOUGH

$$\begin{aligned} \int_0^T f(t)\,\mathrm{d}X_t &= \int_0^T \mathrm{P}^{-1}[f](t)\,\mathrm{d}\hat{X}_t, \\ \int_0^T \hat{g}(t)\,\mathrm{d}\hat{X}_t &= \int_0^T \mathrm{P}[\hat{g}](t)\,\mathrm{d}X_t. \end{aligned}$$

Outline

Canonical Representation for Prediction-Invertible Processes

The next (our main) theorem follows basically by rewriting

$$\int_0^T \mathbf{g}(t)\,\mathrm{d}X_t = \int_0^T \mathrm{P}^{-1}[\mathbf{g}](t)\,\mathrm{d}\hat{X}(t) = \mathbf{0}.$$

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Theorem (Canonical Representation)

Let X be prediction-invertible Gaussian process. Then

$$X_t^{\mathbf{g}} = X_t - \int_0^t \int_s^t p^{-1}(t,u)\mathrm{P}\left[\hat{\ell}^*_{\hat{\mathbf{g}}}(u,\cdot)\right](s)\,\mathrm{d}\langle \hat{X}\rangle_u\,\mathrm{d}X_s,$$

where $\hat{g} = \mathrm{P}^{-1}[g]$ and

$$\hat{\ell}^*_{\hat{\mathbf{g}}}(u,v) = |\langle\!\langle \hat{\mathbf{g}} \rangle\!\rangle^{\hat{X}}|(u)\hat{\mathbf{g}}^{\top}(u)(\langle\!\langle \hat{\mathbf{g}} \rangle\!\rangle^{\hat{X}})^{-1}(u)\,\frac{\hat{\mathbf{g}}(v)}{|\langle\!\langle \hat{\mathbf{g}} \rangle\!\rangle^{\hat{X}}|(v)}.$$

Canonical Representation for Prediction-Invertible Processes

Proof.

In the prediction-martingale level we have

$$\mathrm{d}\hat{X}^{\hat{\mathbf{g}}}_s = \mathrm{d}\hat{X}_s - \int_0^s \hat{\ell}^*_{\hat{\mathbf{g}}}(s,u)\,\mathrm{d}\hat{X}_u\,\mathrm{d}\langle\hat{X}\rangle_s.$$

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Now, by operating with p^{-1} and P we get

$$\begin{aligned} X^{\mathbf{g}}_t &= X_t - \int_0^t p^{-1}(t,s)\left(\int_0^s \hat{\ell}^*_{\hat{\mathbf{g}}}(s,u)\,\mathrm{d}\hat{X}_u\right)\mathrm{d}\langle\hat{X}\rangle_s \\ &= X_t - \int_0^t p^{-1}(t,s)\int_0^s \mathrm{P}\left[\hat{\ell}^*_{\hat{\mathbf{g}}}(s,\cdot)\right](u)\,\mathrm{d}X_u\,\mathrm{d}\langle\hat{X}\rangle_s. \end{aligned}$$

Finally, the claim follows by using the Fubini's theorem. □

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– The End –

Thank you for your attention!