Transfer Principle for nth-Order Fractional Brownian Motion with Applications to Prediction and Equivalence in Law

Tommi Sottinen

University of Vaasa

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References

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Abstract

The *n*th-order fractional Brownian motion (*n*-fBm) was introduced by Perrin et al. (2001). It is the self-similar Gaussian process with Hurst index $H \in (n-1, n)$ having *n*th-order stationary increments. We provide a transfer principle for the *n*-fBm, i.e., we construct a Brownian motion (Bm) from it and then represent the *n*-fBm by using the Bm in a non-anticipative way. By using this representation, we provide the prediction formula for the *n*-fBm and also a representation formula for all the Gaussian processes that are equivalent in law to the *n*-fBm.

The talk is based on Sottinen and Viitasaari (2018) that will appear in the special issue of the journal ''Theory of Probability and Mathematical Statistics" which will be devoted to special topic: Fractality and Fractionality.

Outline

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Self-Similar Gaussian Processes

Recall that a stochastic process $X = (X(t))_{t \geq 0}$ is SELF-SIMILAR with *Hurst-index* $H > 0$, if

$$X(at) \stackrel{\text{law}}{=} a^H X(t), \qquad t \geq 0,$$

for all $a > 0$.

If, in addition, the process X has stationary increments, and is Gaussian, then we see that (up-to a multiplicative constant)

$$\mathbf{E}[X(t)X(s)] = \frac{1}{2}\left[|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right].$$

Such a process is called the FRACTIONAL BROWNIAN MOTION.

Self-Similar Gaussian Processes

For fractional Brownian motion we must have $H \in (0, 1)$ (Actually, $H = 0$ and $H = 1$ are possible, but unnatural.)

Indeed, for fractional Brownian motion $X = B_H$ we have

$$\mathbf{E}[|B_H(t) - B_H(s)|] \leq C_\varepsilon |t - s|^{H-\varepsilon},$$

which is for Gaussian processes, by Azmoodeh et al. (2014) necessary and sufficient for γ-Hölder continuity of all $\gamma < H$.

Now note that for $H > 1$ the H-Hölder continuity means constancy.

A Remark on Self-Similar Gaussian Processes

By Yazigi (2015) a Gaussian process X (that is PURELY NON-DETERMINISTIC) is H-self-similar if and only if there exists a Brownian motion W and a function $F \in L^2([0,1])$ such that

$$X(t) = \int_0^t t^{H-\frac{1}{2}} F\left(\frac{s}{t}\right) \mathrm{d}W(s).$$

This way one can construct (non stationary increment) Gaussian processes with arbitrary self-similarity index $H > 0$.

Outline

Fractional Brownian Motion

The fractional Brownian motion B_H is connected to the standard Brownian motion W via Molchan–Golosov (1969) representations:

$$\begin{aligned} B_H(t) &= \int_0^t k_H(t,s)\,\mathrm{d}W(s), \\ W(t) &= \int_0^t k_H^{(-1)}(t,s)\,\mathrm{d}B_H(s). \end{aligned}$$

Here the first integral is standard Wiener integral. The second integral is so-called abstract Wiener integral introduced after a few slides. The constructions go as follows: B_H is given and W is constructed from it, and then B_H is represented via W.

Fractional Brownian Motion

$$
\begin{aligned}
k_H(t,s) &= d_H \left[\left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} \right. \\
&\quad \left. - \left(H - \frac{1}{2}\right) s^{\frac{1}{2}-H} \int_s^t z^{H-\frac{3}{2}} (z-s)^{H-\frac{1}{2}} \, \mathrm{d}z \right] \\
k_H^{(-1)} &= \frac{1}{d_H} \left[\left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{\frac{1}{2}-H} \right. \\
&\quad \left. - \left(H - \frac{1}{2}\right) s^{\frac{1}{2}-H} \int_s^t z^{H-\frac{3}{2}} (z-s)^{\frac{1}{2}-H} \, \mathrm{d}z \right] \\
d_H &= \sqrt{\frac{2H\Gamma(\frac{3}{2}-H)}{\Gamma(H+\frac{1}{2})\Gamma(2-2H)}}.
\end{aligned}
$$

Fractional Brownian Motion

The (fractional) Isonormal Gaussian Process is the centered Gaussian family $(B_H(h), h \in \mathcal{H}_H)$, where the Hilbert space $\mathcal{H}_H = \mathcal{H}_H([0,T])$ is generated by the covariance

$$r_H(t,s) = \frac{1}{2}\left[|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right]$$

as follows:

1. indicators $\mathbf{1}_t := \mathbf{1}_{[0,t)}$, $t \leq T$, belong to $\mathcal{H}_H$.
2. $\mathcal{H}_H$ is endowed with the inner product $\langle \mathbf{1}_t, \mathbf{1}_s \rangle_{\mathcal{H}_H} := r_H(t,s)$,

and the centered Gaussian family is then defined by the covariance

$$\mathbf{E}\left[B_H(h)B_H(\tilde{h})\right] = \langle h, \tilde{h} \rangle_{\mathcal{H}_H}.$$

Fractional Brownian Motion

The (fractional) Abstract Wiener Integral for $h \in \mathcal{H}_H$ is

$$\int_0^T h(t)\,\mathrm{d}B_H(t) = B_H(h).$$

The kernel representation of the fractional Brownian motion can be used to provide a transfer principle for Wiener integrals. Set

$$k_H^* \mathbf{1}_t = k_H(t,\cdot)$$

and extend linearly and close in $\mathcal{H}_H$.

(Actually, for nice functions $f \in \mathcal{H}_H$ we have

$$(k_H^* f)(t) = k_H(T,t)f(t) + \int_t^T [f(u) - f(t)]\,\frac{\partial k_H(u,t)}{\partial u}\,\mathrm{d}u.$$

)

Fractional Brownian Motion

We have the following transfer principle

$$\begin{aligned}\int_0^T f(t)\,\mathrm{d}B_H(t) &= \int_0^T (k_H^* f)(t)\,\mathrm{d}W(t),\\ \int_0^T g(t)\,\mathrm{d}W(t) &= \int_0^T {k_H^*}^{-1}(t)\,\mathrm{d}B_H(t).\end{aligned}$$

Moreover,

$$\begin{aligned}k_H(t,s) &= (k_H^* \mathbf{1}_t)(s),\\ k_H^{-1}(t,s) &= ({k_H^*}^{-1}\mathbf{1}_t)(s).\end{aligned}$$

Outline

nth Order Fractional Brownian Motion

Let $H > 0$ non-integer, and n such that $H \in (n-1, n)$.

The nth Order Fractional Brownian motion $B_H^{(n)}$ is the centered Gaussian process such that it is H-self-similar and n-stationary (i.e., nth increments are stationary).

Equivalently $B_H^{(n)}$ is defined via recursion:

$$\begin{aligned} B_H^{(n)}(t) &= B_H(t), \qquad \text{if } H \in (0,1), \\ B_H^{(n)}(t) &= \int_0^t B_{H-1}^{(n-1)}(s)\,\mathrm{d}s, \qquad \text{otherwise.} \end{aligned}$$

nTH ORDER FRACTIONAL BROWNIAN MOTION
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Set

$$\begin{aligned} k_H^{(1)}(t,u) &= k_H(t,u), \\ k_H^{(n)}(t,u) &= \int_u^t k_{H-1}^{(n-1)}(s,u)\,\mathrm{d}s. \end{aligned}$$

Then

$$B_H^{(n)}(t) = \int_0^t k_H^{(n)}(t,u)\,\mathrm{d}W(u)$$

defines an nth order fractional Brownian motion. Moreover, the Brownian motion W can be recovered from $B_H^{(n)}$ by

$$W(t) = \int_0^t k_{H-n+1}^{(-1)}(t,u)\,\mathrm{d}\,\frac{\mathrm{d}^{n-1}}{\mathrm{d}u^{n-1}} B_H^{(n)}(u).$$

In particular, the filtrations of W and $B_H^{(n)}$ coincide.

nth Order Fractional Brownian Motion

Define a Hilbert space $\mathcal{H}_H^{(n)} = \mathcal{H}_H^{(n)}([0, T])$ such that:

1. indicators $\mathbf{1}_t := \mathbf{1}_{[0,t)}$, $t \leq T$, belong to $\mathcal{H}_H^{(n)}$.
2. $\mathcal{H}_H^{(n)}$ is endowed with the inner product $\langle \mathbf{1}_t, \mathbf{1}_s \rangle_{\mathcal{H}_H^{(n)}} := r_H^{(n)}(t, s)$, where $r_H^{(n)}$ is the covariance of the nth order fractional Brownian motion.

Then

$$\int_0^T f(s)\,\mathrm{d}B_H^{(n)}(s) = B_H^{(n)}(f)$$

is a centered Gaussian random variable, and for $f, g \in \mathcal{H}_H^{(n)}$

$$\mathbf{E}\left[\int_0^T f(s)\,\mathrm{d}B_H^{(n)}(s) \int_0^T g(s)\,\mathrm{d}B_H^{(n)}(s)\right] = \langle f, g \rangle_{\mathcal{H}_H^{(n)}}.$$

nth Order Fractional Brownian Motion

Define an operator $k^{(n)*} : \mathcal{H}_H^{(n)} \mapsto L^2([0,T])$ by linearly extending

$$k_H^{(n)*}\mathbf{1}_t = k_H^{(n)}(t,\cdot).$$

Then $k_H^{(n)*}$ provides an isometry between $\mathcal{H}_H^{(n)}$ and $L^2([0,T])$. Moreover, for any $f \in \mathcal{H}_H^{(n)}$ we have

$$\int_0^T f(u)\,\mathrm{d}B_H^{(n)}(u) = \int_0^T \left(k_H^{(n)*}f\right)(u)\,\mathrm{d}W(u).$$

Furthermore, if $n \geq 2$, then $L^2([0,T]) \subset \mathcal{H}_H^{(n)}$, and for any $f \in L^2([0,T])$ we have

$$\left(k_H^{(n)*}f\right)(u) = \int_0^T f(t)k_{H-1}^{(n-1)}(t,u)\,\mathrm{d}t.$$

Gaussian processes are either singular or equivalent in law. By Hitsuda (1968) a Gaussian process $\tilde{W} = (\tilde{W}(t), t \in [0, T])$, is equivalent in law to a Brownian motion $W = (W(t), t \in [0, T])$ if and only if there there exists a Volterra kernel $b \in L^2([0, T]^2)$ and a function $a \in L^2([0, T])$ such that

$$\tilde{W}(t) = W(t) - \int_0^t \int_0^s b(s, u)\mathrm{d}W(u)\mathrm{d}s + \int_0^t a(s)\,\mathrm{d}s.$$

Here the Brownian motion W is constructed from $\tilde{W}$ as

$$W(t) = \tilde{W}(t) - \int_0^t \int_0^s b^*(s, u)(\mathrm{d}\tilde{W}(u) - a(u)\mathrm{d}u)\mathrm{d}s - \int_0^t a(s)\,\mathrm{d}s.$$

where b^* is the unique resolvent Volterra kernel of b solving the equation

$$\int_s^t b^*(t, u)b(u, s)\mathrm{d}u = b^*(t, s) + b(t, s) = \int_s^t b(t, u)b^*(u, s)\mathrm{d}u.$$

The log-likelihood ratio of model $\tilde{W}$ over W is

$$\begin{aligned} \ell(t) &= \log \frac{\mathrm{d}\tilde{\mathbf{P}}}{\mathrm{d}\mathbf{P}}\Big|\mathcal{F}_t \\ &= \int_0^t \left[\int_0^s b(s,u)\,\mathrm{d}W(u) + a(s)\right] \mathrm{d}W(s) \\ &\quad -\frac{1}{2}\int_0^t \left[\int_0^s b(s,u)\,\mathrm{d}W(u) + a(s)\right]^2 \mathrm{d}s. \qquad (*) \end{aligned}$$

Consider then a Gaussian process $\tilde{B}_H^{(n)} = (\tilde{B}_H^{(n)}(t), t \in [0, T])$. This process is equivalent to the nth order fractional Brownian motion if and only if the process

$$\tilde{W}(t) = \int_0^t k_{H-n+1}^{(-1)}(t,u)\,\mathrm{d}\,\frac{\mathrm{d}^{n-1}}{\mathrm{d}u^{n-1}}\tilde{B}_H^{(n)}(u). \qquad (**)$$

is equivalent to a Brownian motion.

Now

$$\begin{aligned}
&\tilde{B}_H^{(n)}(t) \\
&= \int_0^t k_H^{(n)}(t,s)\mathrm{d}\tilde{W}(s) \\
&= B_H^{(n)}(t) - \int_0^t k_H^{(n)}(t,s)\int_0^s b(s,u)\,\mathrm{d}W(u)\,\mathrm{d}s \\
&\quad + \int_0^t k_H^{(n)}(t,s)a(s)\,\mathrm{d}s. \qquad (***)
\end{aligned}$$

Note that the integrals above exist, since the integrands are square integrable.

A Gaussian process $\tilde{B}_H^{(n)}$ is equivalent in law to an nth order fractional Brownian motion on $[0, T]$ if and only if there exists a Volterra kernel $b \in L^2([0, T]^2)$ and a function $a \in L^2([0, T])$ such that $\tilde{B}_H^{(n)}$ admits the representation (***), where $\tilde{W}$ is constructed from $\tilde{B}_H^{(n)}$ by (**) and W is a Brownian motion connected to $\tilde{W}$ via Hitsuda representation. The log-likelihood ratio of $\tilde{B}_H^{(n)}$ over $B_H^{(n)}$ is given by (*).

The regular conditional distribution of the nth order fractional Brownian motion $B_H^{(n)} = (B_H^{(n)}(t), t \in [0, T])$ conditioned on the information $\mathcal{F}_{B_H^{(n)}}(u) = \sigma\{B_H^{(n)}(v), v \leq u\}$, is a Gaussian process with random mean $\hat{B}_H^{(n)}(\cdot|u)$ given by

$$\hat{B}_H^{(n)}(t|u) = B_H^{(n)}(u) + \int_0^u \left[k_H^{(n)}(t,v) - k_H^{(n)}(u,v)\right] \mathrm{d}W(v)$$

and a deterministic covariance $\hat{r}_H^{(n)}(\cdot,\cdot|u)$ given by

$$\hat{r}_H^{(n)}(t,s|u) = r_H^{(n)}(t,s) - \int_0^u k_H^{(n)}(t,v)k_H^{(n)}(s,v)\,\mathrm{d}v.$$

Proof of prediction law: By the Gaussian correlation theorem the conditional law is Gaussian with mean

$$t \mapsto \mathbf{E}\left[B_H^{(n)}(t) \,\middle|\, \mathcal{F}_{B_H^{(n)}}(u) \right]$$

and the covariance function

$$\begin{aligned}(t,s) \mapsto\ & \hat{r}_H^{(n)}(t,s|u) \\ = \ & \mathbf{E}\left[\left(B_H^{(n)}(t) - \hat{B}_H^{(n)}(t|u) \right) \left(B_H^{(n)}(s) - \hat{B}_H^{(n)}(s|u) \right) \,\middle|\, \mathcal{F}_{B_H^{(n)}}(u) \right].\end{aligned}$$

$$
\begin{aligned}
&\hat{B}_H^{(n)}(t|s) \\
&= \mathbf{E}\left[B_H^{(n)}(t) \,\middle|\, \mathcal{F}_{B_H^{(n)}}(u) \right] \\
&= \mathbf{E}\left[B_H^{(n)}(t) \,\middle|\, \mathcal{F}_W(u) \right] \\
&= \mathbf{E}\left[\int_0^t k_H^{(n)}(t,v)\,\mathrm{d}W(v) \,\middle|\, \mathcal{F}_W(u) \right] \\
&= \mathbf{E}\left[\int_0^u k_H^{(n)}(t,v)\,\mathrm{d}W(v) + \int_u^t k_H^{(n)}(t,v)\,\mathrm{d}W(v) \,\middle|\, \mathcal{F}_W(u) \right] \\
&= \mathbf{E}\left[\int_0^u k_H^{(n)}(t,v)\,\mathrm{d}W(v) \,\middle|\, \mathcal{F}_W(u) \right] + \mathbf{E}\left[\int_u^t k_H^{(n)}(t,v)\,\mathrm{d}W(v) \right] \\
&= \int_0^u k_H^{(n)}(t,v)\,\mathrm{d}W(v).
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\hat{B}_H^{(n)}(t|u) \\
&= \int_0^u k_H^{(n)}(t,v)\,\mathrm{d}W(v) \\
&= \int_0^u k_H^{(n)}(u,v)\,\mathrm{d}W(v) + \int_0^u \left[k_H^{(n)}(t,v) - k_H^{(n)}(u,v)\right]\,\mathrm{d}W(v) \\
&= B_H^{(n)}(u) + \int_0^u \left[k_H^{(n)}(t,v) - k_H^{(n)}(u,v)\right]\,\mathrm{d}W(v).
\end{aligned}
$$

$$
\begin{aligned}
&\hat{r}_H^{(n)}(t,s|u)\\
&= \mathbf{E}\left[\left(B_H^{(n)}(t)-\hat{B}_H^{(n)}(t|u)\right)\left(B_H^{(n)}(s)-\hat{B}_H^{(n)}(s|u)\right)\,\middle|\,\mathcal{F}_{B_H^{(n)}}(u)\right]\\
&= \mathbf{E}\left[\int_u^t k_H^{(n)}(t,v)\,\mathrm{d}W(v)\int_u^s k_H^{(n)}(s,v)\,\mathrm{d}W(v)\,\middle|\,\mathcal{F}_W(u)\right]\\
&= \int_u^{\min(t,s)} k_H^{(n)}(t,v)k_H^{(n)}(t,v)\,\mathrm{d}v.
\end{aligned}
$$

Note that

$$
r_H^{(n)}(t,s)=\int_0^{\min(t,s)} k_H^{(n)}(t,u)k_H^{(n)}(s,u)\,\mathrm{d}u,
$$

and the proof is completed. □

Thank You for listening!

Any questions or comments?