

INTEGRATION-BY-PARTS CHARACTERIZATIONS OF GAUSSIAN PROCESSES

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ABSTRACT

The Stein's lemma characterizes the Gaussian distribution via an integration-by-parts formula.

We show that a similar integration-by-parts formula characterizes a wide class of Gaussian processes, the so-called Gaussian Fredholm processes.

OUTLINE

- 1 STEIN'S (MULTIVARIATE) LEMMA
- 2 FREDHOLM REPRESENTATION
- 3 PATHWISE MALLIAVIN DIFFERENTIATION
- 4 STRONG FORM INTEGRATION-BY-PARTS
CHARACTERIZATION
- 5 WEAK FORM INTEGRATION-BY-PARTS
CHARACTERIZATION

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STEIN'S (MULTIVARIATE) LEMMA

STEIN'S LEMMA, a.k.a. the INTEGRATION-BY-PARTS CHARACTERIZATION, states that a random variable X is standard normal if and only if

$$\mathbb{E} [Xf'(X)] = \mathbb{E} [f''(X)]$$

for all smooth and bounded enough $f: \mathbb{R} \rightarrow \mathbb{R}$.

MULTIVARIATE STEIN'S LEMMA states that $X = (X_1, \dots, X_d)$ is centered Gaussian with covariance R if and only if

$$\mathbb{E} \left[\sum_{i=1}^d X_i \frac{\partial}{\partial x_i} f(X) \right] = \mathbb{E} \left[\sum_{i=1}^d \sum_{j=1}^d R_{ij} \frac{\partial^2}{\partial x_i \partial x_j} f(X) \right]$$

for all smooth and bounded enough $f: \mathbb{R}^d \rightarrow \mathbb{R}$

STEIN'S (MULTIVARIATE) LEMMA

Let $X = (X_t)_{t \in [0,1]}$ be a centered process with covariance R . The Multivariate Stein's Lemma suggests us to guess (**WRONGLY!**) that X is Gaussian if and only if

$$\mathbb{E} \left[\int_0^1 X_t D_t f(X) dt \right] = \mathbb{E} \left[\int_0^1 \int_0^1 R(t,s) D_{t,s}^2 f(X) ds dt \right],$$

where D is some kind of **MALLIAVIN DERIVATIVE**.

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FREDHOLM REPRESENTATION

THE THEOREM

THEOREM (FREDHOLM REPRESENTATION)

Let $X = (X_t)_{t \in [0,1]}$ be a separable centered Gaussian process. Then there exists a kernel $K \in L^2 \times L^2 = L^2([0, 1]^2)$ and a Brownian motion $W = (W_t)_{t \geq 0}$ such that

$$X_t = \int_0^1 K(t, s) dW_s$$

if and only if the covariance R of X satisfies the trace condition

$$\int_0^T R(t, t) dt < \infty.$$

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PATHWISE MALLIAVIN DIFFERENTIATION

Let $\mathcal{C}_p^\infty(\mathbb{R}^n)$ denote the space of all polynomially bounded functions with polynomially bounded partial derivatives of all orders. Consider functionals $f: L^2 \rightarrow \mathbb{C}$ of the form

$$f(x) = g(z_1, \dots, z_n),$$

where $n \in \mathbb{N}$ and $g \in \mathcal{C}_p^\infty(\mathbb{R}^n)$, and

$$z_k = \int_0^1 e_k(t) dx(t)$$

for some elementary functions $e_k \in \mathcal{E}$. For such f we write $f \in \mathcal{S}$. We call the elements of class \mathcal{S} the **SMOOTH** functionals. The **PATHWISE MALLIAVIN DERIVATIVE** of such $f \in \mathcal{S}$ is

$$D_t f(x) = \sum_{k=1}^n \frac{\partial}{\partial z_k} g(z_1, \dots, z_n) e_k(t).$$

PATHWISE MALLIAVIN DIFFERENTIATION

More generally, by iteration for every $m \in \mathbb{N}$, the pathwise Malliavin derivative of order m is defined as follows: for every $t_1, \dots, t_m \in [0, 1]$,

$$D_{t_m, \dots, t_1}^m f(x) = \sum_{1 \leq k_1, \dots, k_m \leq n} \frac{\partial^m}{\partial z_{k_1} \dots \partial z_{k_m}} g(z_{k_1}, \dots, z_{k_n}) (e_{k_1} \otimes \dots \otimes e_{k_m})(t_1, \dots, t_m).$$

REMARK

Let $f \in \mathcal{S}$ and $y \in L^2$. Then

$$\langle \nabla f(x), ly \rangle_{L^2} = \langle Df(x), y \rangle_{L^2}$$

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STRONG FORM INTEGRATION-BY-PARTS CHARACTERIZATION

Let K^* extend linearly the relation $K^*1_t(s) = K(t, s)$, where $1_t = 1_{[0,t)}$.

THEOREM

Let $K \in L^2 \times L^2$. The co-ordinate process $X: \Omega \rightarrow L^2$ is centered Gaussian with Fredholm kernel K if and only if

$$E[X_t D_t f(X)] = E \left[\int_0^1 K(t, s) K^* [D_t^2, \cdot f(X)](s) ds \right]$$

for all $t \in [0, 1]$ and $f \in \mathcal{S}$.

STRONG IBP, PROOF OF IF PART

Suppose the co-ordinate process $X: \Omega \rightarrow L^2$ satisfies the Strong IBP Formula. We begin by considering the covariance function of X , which will justify the use of the Fubini theorem later and make a tedious variance calculations unnecessary. For this, take $f(X) = \frac{1}{2}X_u^2$ for some $u \in [0, 1]$. Then $f \in \mathcal{S}$. We have $D_t f(X) = X_u 1_u(t)$ and $D_{t,s}^2 f(X) = 1_u(s)1_u(t)$. Consequently,

$$\begin{aligned} \mathbb{E}[X_t X_u] 1_u(t) &= \mathbb{E} \left[\int_0^1 K(t, s) K^* [1_u](s) ds \right] 1_u(t) \\ &= \int_0^1 K(t, s) K(u, s) ds 1_u(t) \\ &= R(t, u) 1_u(t). \end{aligned}$$

This shows that X has the covariance function R given by the Fredholm kernel K .

STRONG IBP, PROOF OF IF PART

In particular, we have

$$\mathbb{E} [X_t^2] = \int_0^1 K(t, s)^2 ds,$$

and since $K \in L^2 \times L^2$, we have $\int_0^1 \mathbb{E} [X_t^2] dt < \infty$ which justifies the use of the Fubini theorem in the rest of the proof. Next we are going to show that any finite linear combination

$$Z = \sum_{k=1}^n a_k (X_{t_k} - X_{t_{k-1}}) = \int_0^1 e(t) dX_t$$

with $e = \sum_k^n a_k \mathbf{1}_{(t_{k-1}, t_k]} \in \mathcal{E}$ is a Gaussian random variable. Now, note that for every θ the complex-valued exponential functional $e^{i\theta Z} = \cos(\theta Z) + i \sin(\theta Z)$ belongs to \mathcal{S} , meaning that the real and imaginary parts both belong to \mathcal{S} .

STRONG IBP, PROOF OF IF PART

Let φ be the characteristic function of Z . Then

$$\begin{aligned}D_t e^{i\theta Z} &= i\theta e(t) e^{i\theta Z}, \\D_{t,s}^2 e^{i\theta Z} &= -\theta^2 e(t) e(s) e^{i\theta Z}.\end{aligned}$$

Hence $E[X_t D_t e^{i\theta Z}] = i\theta e(t) E[X_t e^{i\theta Z}]$. Also, by a direct application of Fubini theorem

$$\begin{aligned}E\left[\int_0^1 K(t,s) K^* \left[D_{t,\cdot}^2 e^{i\theta Z}\right](s) ds\right] \\&= -E\left[\int_0^1 K(t,s) K^* \left[\theta^2 e(t) e(\cdot) e^{i\theta Z}\right](s) ds\right] \\&= -\theta^2 e(t) E\left[\int_0^1 K(t,s) K^* \left[e(\cdot) e^{i\theta Z}\right](s) ds\right] \\&= -\theta^2 e(t) \int_0^1 K(t,s) e^*(s) ds E\left[e^{i\theta Z}\right],\end{aligned}$$

STRONG IBP, PROOF OF IF PART

where we have denoted $e^* = K^* e$. Consequently, the Strong IBP Formula yields

$$i E \left[X_t e^{i\theta Z} \right] = -\theta \int_0^1 K(t, s) e^*(s) ds \varphi(\theta).$$

By Fubini theorem justified by the covariance computation, we also have

$$\varphi'(\theta) = E \left[iZ e^{i\theta Z} \right].$$

Thus we obtain that $\varphi'(\theta) = -c\theta \varphi(\theta)$, where we have denoted

$$c = \int_0^1 \left(\sum_{k=1}^n a_k (K(t_k, s) - K(t_{k-1}, s)) e^*(s) \right) ds < \infty.$$

This implies that $\varphi(\theta) = e^{-\frac{1}{2}c\theta^2}$, and since φ is a characteristic function, $c > 0$. Consequently, Z is a centered Gaussian random variable with variance c .

STRONG IBP, PROOF OF ONLY IF PART

Since the co-ordinate process $X: \Omega \rightarrow L^2$ is Gaussian, we have the full power of Malliavin calculus at our disposal.

In particular, we can use Malliavin IBP Formula

$$E[FG] = E[F]E[G] + E[\langle DF, -DL^{-1}G \rangle_{\mathcal{H}}].$$

with $F = D_t f(X)$ and $G = X_t$. Since $E[X_t] = 0$, we obtain

$$E[X_t D_t f(X)] = E[\langle D_{t,\cdot}^2 f(X), -DL^{-1}X_t \rangle_{\mathcal{I}}]$$

But $-DL^{-1}X_t = 1_t$ and K^* is an isometry between \mathcal{I} and L^2 . Therefore, by noticing that $K^*1_t(s) = K(t, s)$, we obtain

STRONG IBP, PROOF OF ONLY IF PART

$$\begin{aligned} E[X_t D_t f(X)] &= E \left[\langle D_{t,\cdot}^2 f(X), 1_t \rangle_{\mathcal{I}} \right] \\ &= E \left[\langle K^* D_{t,\cdot}^2 f(X), K^* 1_t \rangle_{L^2} \right] \\ &= E \left[\int_0^1 K^* [D_{t,\cdot}^2 f(X)](s) K^* 1_t(s) ds \right] \\ &= E \left[\int_0^1 K^* [D_{t,\cdot}^2 f(X)](s) K(t, s) ds \right] \end{aligned}$$

showing the claim.

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WEAK FORM INTEGRATION-BY-PARTS CHARACTERIZATION

By using Fubini to the Strong IBP Formula, we obtain

THEOREM (WEAK INTEGRATION-BY-PARTS CHARACTERIZATION)

Let $K \in L^2 \times L^2$. Assume that the co-ordinate process $X: \Omega \rightarrow L^2$ satisfies $X \in L^2(dt \otimes P)$, i.e.

$$\int_0^1 \mathbb{E} [X_t^2] dt < \infty.$$

Then X is centered Gaussian with the Fredholm kernel K if and only if

$$\mathbb{E} \left[\int_0^1 X_t D_t f(X) dt \right] = \mathbb{E} \left[\int_0^1 \int_0^1 K(t,s) K^* [D_{t,\cdot}^2 f(X)](s) ds dt \right]$$

for all $f \in \mathcal{S}$.

Thank you for listening!
Any questions?