Gaussian Fredholm Processes

Tommi Sottinen
University of Vaasa, Finland

(Based on a joint work with Lauri Viitasaari, Aalto University, Finland)

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Abstract

MOTTO: Gaussian processes are difficult, Brownian motion is easy.

We show that every separable Gaussian process with integrable variance function admits a Fredholm representation with respect to a Brownian motion.

We extend the Fredholm representation to a transfer principle and develop stochastic analysis by using it. In particular, we prove an Itô formula that is, as far as we know, the most general Skorohod-type Itô formula for Gaussian processes so far.

Finally, we give applications to equivalence in law and series expansions of Gaussian processes.

Outline

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Theorem (Fredholm Representation)

Let $X = (X_t)_{t\in[0,T]}$ be a separable centered Gaussian process. Then there exists a kernel $K_T \in L^2([0,T]^2)$ and a Brownian motion $W = (W_t)_{t\geq 0}$, independent of T, such that

$$X_t = \int_0^T K_T(t,s)\,\mathrm{d}W_s$$

if and only if the covariance R of X satisfies the trace condition

$$\int_0^T R(t,t)\,\mathrm{d}t < \infty.$$

- The Fredholm Kernel K_T usually depends on T even if R does not.
- K_T may be assumed to be symmetric.
- K_T is unique in the sense that if there is another representation with kernel $\tilde{K}_T$, then $\tilde{K}_T = UK_T$ for some unitary operator U on $L^2([0,T])$.
- The Fredholm Representation Theorem holds also for the parameter space $\mathbb{R}_+$, but the trace condition seldom holds, i.e. typically
$$\int_0^\infty R(t,t)\,\mathrm{d}t = \infty.$$
- If the covariance R is degenerate, one needs to extend the probability space to carry the Brownian motion.

- K_T (operator) can be constructed from R_T (operator) as the unique positive symmetric square-root, i.e. the operator K_T is a limit of polynomials:

$$K_T = \lim_{n\to\infty} P_n(R_T).$$

- The positive symmetric square-root is different from the Cholesky square-root. Indeed, the Cholesky square-root would correspond the Volterra representation

$$X_t = \int_0^t K(t,s)\,\mathrm{d}W_s.$$

  The Volterra representation does not hold for Gaussian processes in general.

Consider a truncated series expansion

$$X_t = \sum_{k=1}^{n} \int_0^t e_k^T(t)\,\mathrm{d}t\;\xi_k,$$

where ξ_k are i.i.d.$\sim N(0,1)$ and e_k^T, $k \in \mathbb{N}$, is an orthonormal basis in $L^2([0,T])$.

X is not PURELY NON-DETERMINISTIC. Consequently, X does not admit Volterra representation.

Choosing e_k^T to be the trigonometric basis, X is a finite-rank approximation of the Karhunen–Loève representation of standard Brownian motion on $[0,T]$. Hence by letting $n \to \infty$ we obtain a standard Brownian motion, and hence a Volterra process.

Let W be the Brownian motion and B the Brownian bridge B.
The orthogonal representation of B is

$$B_t = W_t - \frac{t}{T} W_T.$$

Thus, B has a Fredholm representation with kernel

$$K_T(t,s) = \mathbf{1}_{[0,t)}(s) - \frac{t}{T}.$$

The canonical representation of the Brownian bridge is

$$B_t = (T-t) \int_0^t \frac{1}{T-s} \, \mathrm{d}W_s.$$

Hence B has also a Volterra representation with kernel

$$K(t,s) = \frac{T-t}{T-s}.$$

By the Mercer's theorem

$$R(t,s) = \sum_{i=1}^{\infty} \lambda_i^T e_i^T(t) e_i^T(s),$$

where $(\lambda_i^T)_{i=1}^{\infty}$ and $(e_i^T)_{i=1}^{\infty}$ are the eigenvalues and the eigenfunctions of the covariance operator

$$R_T f(t) = \int_0^T f(s) R(t,s) \, \mathrm{d}s.$$

Moreover, $(e_i^T)_{i=1}^{\infty}$ is an orthonormal system on $L^2([0,T])$.

Since R_T is a covariance-operator, it admits a square-root operator K_T. By the trace condition R_T is trace-class, and hence K_T is Hilbert-Schmidt. Thus, K_T admits a Kernel.

Indeed,

$$K_T(t,s) = \sum_{i=1}^{\infty} \sqrt{\lambda_i^T} e_i^T(t) e_i^T(s).$$

Now K_T is obviously symmetric and we have

$$R(t,s) = \int_0^T K_T(t,u) K_T(s,u) \, \mathrm{d}u$$

from which the Fredholm Representation follows by enlarging the probability space, if needed.

This shows that the Fredholm representation holds in law. To make it hold in $L^2(\Omega)$ one can construct the Brownian motion associated with X by using $L^2(\Omega)$ isometries by starting from an i.i.d.$\sim N(0,1)$ sequence. $\square$

Outline

The adjoint operator Γ^* of a kernel $\Gamma \in L^2([0, T]^2)$ is defined by linearly extending the relation

$$\Gamma^* \mathbf{1}_{[0,t)} = \Gamma(t, \cdot).$$

Remark

If $\Gamma(\cdot, s)$ is of bounded variation for all s and f is nice enough, then

$$\Gamma^* f(s) = \int_0^T f(t)\, \Gamma(\mathrm{d}t, s).$$

Transfer Principle

for Malliavin Derivatives and Skorohod Integrals

Theorem (Transfer Principle)

Let X be Gaussian Fredholm process with kernel K_T. Let D_T, δ_T, D_T^W and δ_T^W be the Malliavin derivative and the Skorohod integral with respect to X and to the Brownian motion W. Then

$$\delta_T = \delta_T^W K_T^* \quad \text{and} \quad K_T^* D_T = D_T^W.$$

Proof: Trivial. □

Theorem (Itô Formula)

Let X be centered Gaussian process with covariance R and let $f \in C^2$. Then

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\,\delta X_s + \frac{1}{2}\int_0^t f''(X_s)\,\mathrm{d}R(s,s),$$

if anything.

Proof: Trivial. $\square$

Outline

Let us show how to use the Fredholm Representation and the Transfer Principle to analyze equivalence of Gaussian laws.

Recall the Hitsuda Representation Theorem: A centered Gaussian process $\tilde{W}$ is equivalent to a Brownian motion W if and only if there exists a Volterra kernel $\ell \in L^2([0, T]^2)$ such that

$$\mathrm{d}\tilde{W}_t = \mathrm{d}W_t + \int_0^t \ell(t,s)\,\mathrm{d}W_s \cdot \mathrm{d}t.$$

Now, let $\tilde{X}$ and X be Gaussian Fredholm processes with

$$\begin{aligned} \tilde{X}_t &= \int_0^T \tilde{K}_T(t,s)\,\mathrm{d}W_s, \\ X_t &= \int_0^T K_T(t,s)\,\mathrm{d}W_s. \end{aligned}$$

Suppose then that $\tilde{X}$ has (also) representation

$$\tilde{X}_t = \int_0^T K_T(t,s)\,\mathrm{d}\tilde{W}_s$$

where $\tilde{W}$ and W are equivalent.

Then, obviously $\tilde{X}$ and X are equivalent.

By plugging in the Hitsuda connection we obtain

$$\tilde{X}_t = \int_0^T \left[K_T(t,s) + \int_s^T K_T(t,u)\ell(u,s)\,\mathrm{d}u\right]\mathrm{d}W_s.$$

Theorem (Equivalence of Laws)

Let X and $\tilde{X}$ be two Gaussian process with Fredholm kernels K_T and $\tilde{K}_T$, respectively. If there exists a Volterra kernel $\ell \in L^2([0,T]^2)$ such that

$$\tilde{K}_T(t,s) = K_T(t,s) + \int_s^T K_T(t,u)\ell(u,s)\,\mathrm{d}u,$$

then X and $\tilde{X}$ are equivalent in law.
If the kernel K_T satisfies appropriate non-degeneracy property, then the condition above is also necessary.

In the same way, as in the case of equivalence of laws, we see that:

Theorem (Series representation)

Let X be a Gaussian Fredholm process with kernel K_T and let φ_k^T, $k \in \mathbb{N}$, be any orthonormal basis in $L^2([0,T])$. Then

$$X_t = \sum_{k=1}^{\infty} \int_0^T K_T(t,s)\varphi_k^T(s)\,\mathrm{d}s \cdot \xi_k,$$

where ξ_k, $k \in \mathbb{N}$, are i.i.d. standard Gaussian random variables. The series above converges in $L^2(\Omega)$; and also almost surely uniformly if and only if X is continuous.

Outline

The Moral of the Story

- In general it is difficult, if not impossible, to construct the K_T satisfying $R_T = K_T^2$ ANALYTICALLY.
- There is an ALGORITHM to construct K_T as a limit of polynomials of R_T: Set

$$Y_0 = 0, \quad Y_1 = \frac{1}{2}(I - R_T), \quad Y_{n+1} = \frac{1}{2}(I - R_T + Y_n^2).$$

  Then

$$K_T = I - Y_\infty,$$

  since

$$Y_\infty = \frac{1}{2}(I - R_T + Y_\infty^2).$$

- WHY NOT START THE MODELING WITH THE FREDHOLM REPRESENTATION!

Outline

An Open Question

It is clear that the Fredholm representation

$$X_t = \int_0^T K_T(t,s)\,\mathrm{d}W_s$$

cannot, in general, be inverted for the Brownian motion W.

But is it possible to find a Gaussian martingale M such that we have both Fredholm and inverse Fredholm representations

$$\begin{aligned} X_t &= \int_0^T K_T(t,s)\,\mathrm{d}M_s, \\ M_t &= \int_0^T K_T^{-1}(t,s)\,\mathrm{d}X_s? \end{aligned}$$

Is it possible to make the relations above Volterra?

Thank you for listening!

Any questions?