Gaussian (Fredholm) Processes

Tommi Sottinen
University of Vaasa, Finland

(Based on a joint work with Lauri Viitasaari, Aalto University, Finland)

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**Motto:** Gaussian processes are difficult, Brownian motion is easy.

We show that every separable Gaussian process with integrable variance function admits a Fredholm representation with respect to a Brownian motion.

We extend the Fredholm representation to a transfer principle and develop stochastic analysis by using it. In particular, we prove an Itô formula that is, as far as we know, the most general Skorohod-type Itô formula for Gaussian processes so far.

Finally, we give applications to equivalence in law and series expansions of Gaussian processes.
Outline

1 Fredholm Representation

2 Transfer Principle

3 Applications

4 The Moral of the Story

5 An Open Question
Outline

1. Fredholm Representation
2. Transfer Principle
3. Applications
4. The Moral of the Story
5. An Open Question
**Theorem (Fredholm Representation)**

Let \( X = (X_t)_{t \in [0, T]} \) be a separable centered Gaussian process. Then there exists a kernel \( K_T \in L^2([0, T]^2) \) and a Brownian motion \( W = (W_t)_{t \geq 0} \), independent of \( T \), such that

\[
X_t = \int_0^T K_T(t, s) \, dW_s
\]

if and only if the covariance \( R \) of \( X \) satisfies the trace condition

\[
\int_0^T R(t, t) \, dt < \infty.
\]
The Fredholm Kernel $K_T$ usually depends on $T$ even if $R$ does not.

$K_T$ may be assumed to be symmetric.

$K_T$ is unique in the sense that if there is another representation with kernel $\tilde{K}_T$, then $\tilde{K}_T = UK_T$ for some unitary operator $U$ on $L^2([0, T])$.

The Fredholm Representation Theorem holds also for the parameter space $\mathbb{R}_+$, but the trace condition seldom holds, i.e. typically

$$\int_0^\infty R(t, t) \, dt = \infty.$$ 

If the covariance $R$ is degenerate, one needs to extend the probability space to carry the Brownian motion.
Fredholm Representation
Some Square-Root Remarks

- $K_T$ (operator) can be constructed from $R_T$ (operator) as the unique positive symmetric square-root, i.e. the operator $K_T$ is a limit of polynomials:

$$K_T = \lim_{n \to \infty} P_n(R_T).$$

- The positive symmetric square-root is different from the Cholesky square-root. Indeed, the Cholesky square-root would correspond the Volterra representation

$$X_t = \int_0^t K(t, s) \, dW_s.$$

The Volterra representation does not hold for Gaussian processes in general.
Consider a truncated series expansion

\[ X_t = \sum_{k=1}^{n} \int_{0}^{t} e_k^T(t) \, dt \, \xi_k, \]

where \( \xi_k \) are i.i.d. \( \sim \mathcal{N}(0, 1) \) and \( e_k^T, k \in \mathbb{N} \), is an orthonormal basis in \( L^2([0, T]) \).

\( X \) is not **purely non-deterministic**. Consequently, \( X \) does not admit Volterra representation.

Choosing \( e_k^T \) to be the trigonometric basis, \( X \) is a finite-rank approximation of the Karhunen–Loève representation of standard Brownian motion on \([0, T]\). Hence by letting \( n \to \infty \) we obtain a standard Brownian motion, and hence a Volterra process.
Fredholm Representation
Example II

Let $W$ be the Brownian motion and $B$ the Brownian bridge $B$. The **ORTHOGONAL REPRESENTATION** of $B$ is

$$B_t = W_t - \frac{t}{T} W_T.$$ 

Thus, $B$ has a Fredholm representation with kernel

$$K_T(t, s) = 1_{[0, t)}(s) - \frac{t}{T}.$$ 

The **CANONICAL REPRESENTATION** of the Brownian bridge is

$$B_t = (T - t) \int_0^t \frac{1}{T - s} \mathrm{d}W_s.$$ 

Hence $B$ has also a Volterra representation with kernel

$$K(t, s) = \frac{T - t}{T - s}.$$
**Fredholm Representation**

**The Proof**

By the Mercer’s theorem

\[
R(t, s) = \sum_{i=1}^{\infty} \lambda_i^T e_i^T(t) e_i^T(s),
\]

where \((\lambda_i^T)_{i=1}^{\infty}\) and \((e_i^T)_{i=1}^{\infty}\) are the eigenvalues and the eigenfunctions of the covariance operator

\[
R_T f(t) = \int_0^T f(s) R(t, s) \, ds.
\]

Moreover, \((e_i^T)_{i=1}^{\infty}\) is an orthonormal system on \(L^2([0, T])\).

Since \(R_T\) is a covariance-operator, it admits a square-root operator \(K_T\). By the trace condition \(R_T\) is trace-class, and hence \(K_T\) is Hilbert-Schmidt. Thus, \(K_T\) admits a Kernel.
Fredholm Representation
The Proof

Indeed,

\[ K_T(t, s) = \sum_{i=1}^{\infty} \sqrt{\lambda_i^T} e_i^T(t)e_i^T(s). \]

Now \( K_T \) is obviously symmetric and we have

\[ R(t, s) = \int_0^T K_T(t, u)K_T(s, u) \, du \]

from which the Fredholm Representation follows by enlarging the probability space, if needed.

This shows that the Fredholm representation holds in law. To make it hold in \( L^2(\Omega) \) one can construct the Brownian motion associated with \( X \) by using \( L^2(\Omega) \) isometries by starting from an i.i.d. \( \sim N(0, 1) \) sequence.
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The adjoint operator $\Gamma^*$ of a kernel $\Gamma \in L^2([0, T]^2)$ is defined by linearly extending the relation

$$\Gamma^* 1_{[0,t]} = \Gamma(t, \cdot).$$

**Remark**

If $\Gamma(\cdot, s)$ is of bounded variation for all $s$ and $f$ is nice enough, then

$$\Gamma^* f(s) = \int_0^T f(t) \Gamma(dt, s).$$
**Theorem (Transfer Principle)**

Let $X$ be Gaussian Fredholm process with kernel $K_T$. Let $D_T$, $\delta_T$, $D^W_T$ and $\delta^W_T$ be the Malliavin derivative and the Skorohod integral with respect to $X$ and to the Brownian motion $W$. Then

$$\delta_T = \delta^W_T K_T^*$$

and

$$K_T^* D_T = D^W_T.$$  

**Proof:** Trivial.
**Theorem (Itô Formula)**

Let $X$ be centered Gaussian process with covariance $R$ and let $f \in C^2$. Then

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) \, \delta X_s + \frac{1}{2} \int_0^t f''(X_s) \, dR(s, s),$$

if anything.

**Proof:** Trivial.
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Let us show how to use the Fredholm Representation and the Transfer Principle to analyze equivalence of Gaussian laws.

Recall the **Hitsuda Representation Theorem**: A centered Gaussian process $\tilde{W}$ is equivalent to a Brownian motion $W$ if and only if there exists a Volterra kernel $\ell \in L^2([0, T]^2)$ such that

$$d\tilde{W}_t = dW_t + \int_0^t \ell(t, s) \, dW_s \cdot dt.$$ 

Now, let $\tilde{X}$ and $X$ be Gaussian Fredholm processes with

$$\tilde{X}_t = \int_0^T \tilde{K}_T(t, s) \, dW_s,$$

$$X_t = \int_0^T K_T(t, s) \, dW_s.$$
Suppose then that $\tilde{X}$ has (also) representation

$$\tilde{X}_t = \int_0^T K_T(t, s) \, d\tilde{W}_s$$

where $\tilde{W}$ and $W$ are equivalent.

Then, obviously $\tilde{X}$ and $X$ are equivalent.

By plugging in the Hitsuda connection we obtain

$$\tilde{X}_t = \int_0^T \left[ K_T(t, s) + \int_s^T K_T(t, u)\ell(u, s) \, du \right] \, dW_s.$$
**Theorem (Equivalence of Laws)**

Let $X$ and $\tilde{X}$ be two Gaussian process with Fredholm kernels $K_T$ and $\tilde{K}_T$, respectively. If there exists a Volterra kernel $\ell \in L^2([0, T]^2)$ such that

$$\tilde{K}_T(t, s) = K_T(t, s) + \int_s^T K_T(t, u)\ell(u, s) \, du,$$

then $X$ and $\tilde{X}$ are equivalent in law.

If the kernel $K_T$ satisfies appropriate non-degeneracy property, then the condition above is also necessary.
In the same way, as in the case of equivalence of laws, we see that:

**Theorem (Series representation)**

Let \( X \) be a Gaussian Fredholm process with kernel \( K_T \) and let \( \varphi_T^k \), \( k \in \mathbb{N} \), be any orthonormal basis in \( L^2([0, T]) \). Then

\[
X_t = \sum_{k=1}^{\infty} \int_0^T K_T(t, s)\varphi_T^k(s) \, ds \cdot \xi_k,
\]

where \( \xi_k \), \( k \in \mathbb{N} \), are i.i.d. standard Gaussian random variables.

The series above converges in \( L^2(\Omega) \); and also almost surely uniformly if and only if \( X \) is continuous.
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The Moral of the Story

- In general it is difficult, if not impossible, to construct the $K_T$ satisfying $R_T = K_T^2$ **analytically**.

- There is an **algorithm** to construct $K_T$ as a limit of polynomials of $R_T$: Set

  $$Y_0 = 0, \quad Y_1 = \frac{1}{2}(I - R_T), \quad Y_{n+1} = \frac{1}{2}(I - R_T + Y_n^2).$$

  Then

  $$K_T = I - Y_\infty,$$

  since

  $$Y_\infty = \frac{1}{2}(I - R_T + Y_\infty^2).$$

- **Why not start the modeling with the Fredholm representation!**
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An Open Question

It is clear that the Fredholm representation

\[ X_t = \int_0^T K_T(t, s) \, dW_s \]

cannot, in general, be inverted for the Brownian motion \( W \).

But is it possible to find a Gaussian martingale \( M \) such that we have both Fredholm and inverse Fredholm representations

\[ X_t = \int_0^T K_T(t, s) \, dM_s, \]
\[ M_t = \int_0^T K_T^{-1}(t, s) \, dX_s? \]

Is it possible to make the relations above Volterra?
Thank you for listening!
Any questions?