Power series series expansions of fractional Brownian motion

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Aim and way

The *fractional Brownian motion* (fBm) is a centred Gaussian process $Z = (Z_t)_{t\in[0,1]}$ with covariance

$$R_H(t,s) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

Here $H \in (0,1)$ is the so-called *Hurst index*.

Aim: We want to represent the fBm as

$$Z_t = \sum_{n=0}^{\infty} \varphi_n(t)\xi_n,$$

where ξ_n's are i.i.d. standard Gaussian.

The convergence will be in $L^2(\Omega)$ and almost sure uniformly in $t \in [0,1]$.

Way: To construct the functions φ_n we take the following route:

$$\begin{array}{ll} \text{Linear space} & \\ \quad\updownarrow & \\ \text{Reproducing kernel Hilbert space} & \\ \quad\updownarrow & \leftarrow \text{Power series expansion} \\ L^2([0,1]) & \end{array}$$

Linear space

Let $X = (X_t)_{t\in[0,1]}$ be a centred Gaussian process. Its *linear space* $\mathcal{H} = \mathcal{H}(X)$ is

$$\mathcal{H} := \mathrm{cl}_{L^2(\Omega)}\,\mathrm{span}\,\{X_t : t \in [0,1]\}\,.$$

$\mathcal{H}$ is a Gaussian Hilbert space. If $\mathcal{H}$ separable, then

$$X_t = \sum_{n=1}^{\infty} \mathbb{E}(X_t\xi_n)\xi_n \qquad \text{in } L^2(\Omega),$$

where $(\xi_n)_{n=1}^{\infty}$ is a CONS in $\mathcal{H}$, i.e ξ_n's are i.i.d standard Gaussian.

The convergence is almost sure for all $t \in [0,1]$ (the martingale convergence theorem).

Problem: Find the coefficient functions $t \mapsto \mathbb{E}(X_t\xi_n)$ (and the corresponding CONS).

Reproducing kernel Hilbert space (RKHS)

Let $R(t,s) = \mathbb{E}(X_t X_s)$. We construct a Hilbert space by expanding the relation

$$\Theta : X_t \mapsto R(t,\cdot).$$

More precisely, set

$$\mathcal{S} := \mathrm{span}\left\{R(t,\cdot) : t \in [0,1]\right\}.$$

Define an inner product on $\mathcal{S}$ by expanding

$$\langle R(t,\cdot), R(s,\cdot)\rangle_{\mathcal{R}} := R(t,s).$$

The *Reproducing kernel Hilbert space* $\mathcal{R} = \mathcal{R}(X) = \mathcal{R}(R)$ is

$$\mathcal{R} := \mathrm{cl}_{\langle\cdot,\cdot\rangle_{\mathcal{R}}}\mathcal{S}.$$

Θ is an isometry from $\mathcal{R}$ to $\mathcal{H}$. If R is continuous then $\mathcal{R}$ (and hence $\mathcal{H}$) is separable.

Reproducing property and series expansion

Let R be continuous. The space $(\mathcal{S}, \langle\cdot,\cdot\rangle_{\mathcal{R}})$ has a *reproducing property*. Let

$$f = \sum_{k=1}^{n} a_k R(s_k, \cdot) \;\in\; \mathcal{S}.$$

Then

$$\begin{aligned} f(t) &= \sum_{k=1}^{n} a_k \langle R(s_k,\cdot), R(t,\cdot)\rangle_{\mathcal{R}} \\ &= \left\langle \sum_{k=1}^{n} a_k R(s_k,\cdot), R(t,\cdot) \right\rangle_{\mathcal{R}} \\ &= \langle f, R(t,\cdot)\rangle_{\mathcal{R}}. \end{aligned}$$

This extends to $\mathcal{R}$ by separability.

Let $(\varphi_n)_{n=0}^{\infty}$ be a CONS in $\mathcal{R}$ then

$$\begin{aligned} R(t,\cdot) &= \sum_{n=0}^{\infty} \langle R(t,\cdot), \varphi_n\rangle_{\mathcal{R}} \varphi_n \\ &= \sum_{n=0}^{\infty} \varphi_n(t)\varphi_n. \end{aligned}$$

Reproducing property and series expansion, cont.

If $(\varphi_n)_{n=0}^{\infty}$ is a CONS in $\mathcal{R}$ then $(\Theta(\varphi_n))_{n=0}^{\infty}$ is a CONS in $\mathcal{H}$, i.e. they are i.i.d. standard Gaussian random variables. So,

$$\begin{aligned} X_t &= \Theta\left(R(t,\cdot)\right) \\ &= \Theta\left(\sum_{n=1}^{\infty} \varphi_n(t)\varphi_n\right) \\ &= \sum_{n=0}^{\infty} \varphi_n(t)\Theta(\varphi_n) \\ &= \sum_{n=0}^{\infty} \varphi_n(t)\xi_n, \end{aligned}$$

where $\xi_n = \Theta(\varphi_n)$ and

$$\varphi_n(t) = \langle R(t,\cdot), \varphi_n \rangle_{\mathcal{R}} = \mathbb{E}(X_t \xi_n).$$

By Itô–Nisio theorem the representation is a.s. uniform in $t \in [0,1]$ iff X is continuous.

Problem: Find a CONS $(\varphi_n)_{n=0}^{\infty}$ of $\mathcal{R}$.

RKHS and $L^2[0,1]$

Suppose that R may be written as

$$R(t,s) = \int_0^1 k(t,x)k(s,x)\,\mathrm{d}x$$

for some Volterra kernel $k \in L^2[0,1]^2$.

So we have an isometry

$$\Psi : L^2[0,1]/\mathrm{Ker}\Psi \to \mathcal{R}$$

by extending the relation

$$\Psi : k(t,\cdot) \mapsto R(t,\cdot),$$

i.e

$$(\Psi f)(t) = \int_0^1 k(t,x)f(x)\,\mathrm{d}x.$$

If Ψ *one-to-one* then $\mathcal{R}$ is isometric to $L^2[0,1]$.

In any case $\mathcal{R}$ (and thus $\mathcal{H}$) is separable.

$L^2[0,1]$ and series expansion

Let Ψ be one-to-one. We have the picture:

$$\begin{array}{ccccc} & \Psi & & \Theta & \\ L^2[0,1] & \longrightarrow & \mathcal{R} & \longrightarrow & \mathcal{H} \\ k(t,\cdot) & \mapsto & R(t,\cdot) & \mapsto & X_t \\ \tilde{\varphi}_n & \mapsto & \varphi_n & \mapsto & \xi_n \end{array}$$

For any CONS $(\varphi_n)_{n=0}^{\infty}$ of $\mathcal{R}$ we had

$$X_t = \sum_{n=1}^{\infty} \varphi_n(t)\xi_n$$

Let $(\tilde{\varphi}_n)_{n=1}^{\infty}$ be any CONS in $L^2[0,1]$ (many examples known). The isometry Ψ yields

$$X_t = \sum_{n=0}^{\infty} \left[\int_0^1 k(t,x)\tilde{\varphi}_n(x)\,\mathrm{d}x \right] \cdot \xi_n,$$

where $\xi_n = (\Theta \circ \Psi)(\tilde{\varphi}_n)$.

So we have a concrete series expansion *if we can calculate the integral* for some CONS.

The case of fBm

For fBm Z with Hurst index H we have

$$R_H(t,s) = \int_0^1 z_H(t,x) z(s,x)\,\mathrm{d}x,$$

where z is the Volterra kernel

$$z_H(t,s) = -c_H s^{\frac{1}{2}-H} \frac{\mathrm{d}}{\mathrm{d}s} \int_s^t x^{H-\frac{1}{2}} (x-s)^{H-\frac{1}{2}}\,\mathrm{d}x$$

and c_H is a normalising constant.

Here Ψ_H is one-to-one. Indeed, there is a "resolvent" kernel z_H^* such that

$$(\Psi_H^{-1} f)(t) = \int_0^1 z_H^*(t,x) f(x)\,\mathrm{d}x.$$

Unfortunately, the kernel z_H is a nasty one: it is not easy to calculate, or even approximate

$$\int_0^1 z_H(t,x) \tilde{\varphi}_n(x)\,\mathrm{d}x$$

for a given function $\tilde{\varphi}_n \in L^2[0,1]$.

$$(\Psi_H x^\beta)(t) = c_{H,\beta} t^{H+\frac{1}{2}+\beta}$$

$$(\Psi_H x^\beta)(t) = \int_0^t z_H(t,s) s^\beta \, \mathrm{d}s$$

$$= -c_H \int_0^t s^{\frac{1}{2}-H} \frac{\mathrm{d}}{\mathrm{d}s} \int_s^t x^{H-\frac{1}{2}} (x-s)^{H-\frac{1}{2}} \, \mathrm{d}x \, s^\beta \, \mathrm{d}s$$

$$= c''_H \int_0^t \int_s^t x^{H-\frac{1}{2}} (x-s)^{H-\frac{1}{2}} \, \mathrm{d}x \, \mathrm{d}s^{\frac{1}{2}-H+\beta}$$

$$= c''_{H,\beta} \int_0^t \int_0^x (x-s)^{H-\frac{1}{2}} s^{\beta-\frac{1}{2}-H} \, \mathrm{d}s \, x^{H-\frac{1}{2}} \, \mathrm{d}x$$

$$= c''_{H,\beta} \int_0^t \int_0^1 (x-xu)^{H-\frac{1}{2}} (xu)^{\beta-\frac{1}{2}-H} \, x\mathrm{d}u \, x^{H-\frac{1}{2}} \, \mathrm{d}x$$

$$= c'_{H,\beta} \int_0^t x^\beta \, x^{H-\frac{1}{2}} \, \mathrm{d}x = c_{H,\beta} \, t^{H+\frac{1}{2}+\beta}$$

Polynomial representation

The *shifted Legendre polynomials*

$$\tilde{\varphi}_n^{\text{poly}}(t) = \sum_{\ell=0}^{n} \sum_{k=0}^{\lfloor \frac{n-\ell}{2} \rfloor} d_{n,\ell,k}\, t^\ell$$

where

$$d_{n,\ell,k} = \frac{(-1)^{n-k-\ell}}{2^{n-\ell}} \binom{n}{k} \binom{2n-2k}{n} \binom{n-2k}{\ell}$$

form a CONS on $L^2[0,1]$. So, we get a series representation of fBm with

$$\varphi_n^{\text{poly}}(t) = t^{H+\frac{1}{2}} \sum_{\ell=0}^{n} \sum_{k=0}^{\lfloor \frac{n-\ell}{2} \rfloor} e_{n,\ell,k,H}\, t^\ell$$

where

$$e_{n,\ell,k,H} = d_{n,\ell,k}\, c_{H,n}.$$

This approach does not seem to be computationally stable.

Trigonometric representation

Since

$$\begin{aligned}\tilde{\varphi}_n^{\text{trig}}(t) &= \sqrt{2}\cos(n\pi t)\\ &= \sqrt{2}\sum_{k=0}^{\infty}(-1)^k\frac{(n\pi t)^{2k}}{(2k)!}\end{aligned}$$

we have a series representation of fBm with

$$\varphi_n^{\text{trig}}(t) = \frac{c_H\Gamma(\frac{1}{2}-H)}{H+\frac{1}{2}}\,t^{H+\frac{1}{2}}\,F_H\Big(-\tfrac{1}{4}(n\pi t)^2\Big)$$

where F_H is the hypergeometric function

$$F_H = {}_3F_4\left(\begin{matrix}\frac{5-2H}{4},\ \frac{1+2H}{4},\ \frac{3-2H}{2} \\ 1,\ \frac{5+2H}{4},\ \frac{1}{2},\ \frac{1}{2}\end{matrix}\ ;\ \cdot\right).$$

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