Option-Pricing without Probability

Good News and Bad News

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IFAM Virtual Seminar
9 September 2020, Liverpool, UK

Abstract

We show, by using the Föllmer interpretation of the Itô integral, that the hedges of financial derivatives are independent of the probabilistic properties of the underlying asset. Also, by assuming a so-called conditional full support condition and rough enough paths on the underlying, we show that there are no reasonable arbitrage opportunities.

The good news are:

1. Stylized facts are irrelevant in option-pricing.
2. Black–Scholes type formulas are correct even when the underlying returns are not Gaussian.
3. Quadratic variation is all that matters, and it has nothing to do with probability.

Abstract

The bad news are:

1. The use of martingale techniques like equivalent martingale measures are dubious.
2. Classical statistical estimation is wrong.
3. Errors in discrete hedging may converge arbitrarily slow.

The talk is based on the articles

1. Bender, C., Sottinen, T. and Valkeila, E. (2008) Pricing by hedging and no-arbitrage beyond semimartingales. Finance and Stochastics 12, 441-468.
2. Bender, C., Sottinen, T. and Valkeila, E. (2011) Fractional processes as models in stochastic finance. Advanced Mathematical Methods for Finance. Series in Mathematical Finance, Springer, pp.75-103.

Reasons against non-semimartingales:

- In stochastic finance one needs integration theory to define self-financing trading strategies.
- The semimartingale theory for integration is well-suited for stochastic finance.
- By Bichteler–Dellacherie–Mokobodsky Theorem the semimartingales are the largest class of integrators that have continuous integrals. Thus one expects arbitrage with non-semimartingales. Indeed, Delbaen and Schachermeyer have proved that there will be arbitrage for non-semimartingales.
- It is an economic axiom that there should be no arbitrage: "There is no such thing as a free lunch".

Reasons for non-semimartingales:

- It is unclear if the Delbaen–Schachermeyer arbitrage is practical.
- Classical models do not fit well to real financial data, and some stylized facts (e.g. long range dependence) are difficult to incorporate into the semimartingale world.
- Even with semimartingales one does not use actual probabilities, but so-called equivalent martingale measures. So, it is important to know how little probability is actually needed.

Outline

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- Let $(B_t)_{t\in[0,T]}$ be the bond, or BANK ACCOUNT. We work in the discounted world:

$$B_t = 1 \qquad \text{for all } t \in [0, T].$$

  The bank account is RISKLESS, i.e. non-random.
- Let $S = (S_t)_{t\in[0,T]}$ be the STOCK. The stock is RISKY, i.e. random.
  1. CLASSICAL ASSUMPTION: S is a (continuous) semimartingale, typically the GEOMETRIC BROWNIAN MOTION.
  2. OUR ASSUMPTION: S is continuous and has QUADRATIC VARIATION a.s. (We elaborate what we mean by quadratic variation later.)

Definition (Option)

Option is simply a real-valued mapping $S \mapsto G(S)$. The asset S is the underlying of the option G.

Example

- $G = (S_T - K)^+$ is a call-option,
- $G = (K - S_T)^+$ is a put-option,
- $G = S_T - K$ is a future.

T is the time of maturity and K is the strike-price.

Definition (Self-financing Trading Strategy)

Trading strategy $\Phi = (\Phi_t)_{t\in[0,T]}$ is an S-adapted stochastic process that tells the units of the underlying asset S the investor has is her portfolio at any time $t \in [0, T]$. The wealth of a self-financing trading strategy Φ satisfies the forward differential (to be defined properly later)

$$\mathrm{d}V_t(\Phi) \; = \; \Phi_t \, \mathrm{d}S_t. \tag{1}$$

Remark

(1) corresponds to the budget constraint

$$V_{t+\Delta t}(\Phi) \; = \; \Phi_t S_{t+\Delta t} + \left(V_t(\Phi) - \Phi_t S_t\right).$$

Replication principle is used to hedge and price options.

Definition (Replication principle)

Let G be an option. Suppose that there is a self-financing trading strategy Φ with wealth $V_t(\Phi)$ at time t such that $G = V_T(\Phi)$ at time T. Then the price of the option G at time t is $V_t(\Phi)$.

Remark

The replication requirement $G = V_T(\Phi)$ can be written as the forward integral (to be defined properly later)

$$G \;=\; V_t(\Phi) + \int_t^T \Phi_s \,\mathrm{d}S_s.$$

Arbitrage is "free lunch": Profit without risk or capital.

Definition (Arbitrage)

Arbitrage is a self-financing trading strategy Φ such that $V_0(\Phi) = 0$, $\mathbf{P}[V_T(\Phi) \geq 0] = 1$, and $\mathbf{P}[V_T(\Phi) > 0] > 0$.

No-arbitrage principle can be used to price options.

Definition (No-Arbitrage Principle)

No-Arbitrage Price of an option is any price that does not induce arbitrage into the market when the option is considered as a new asset.

Remark

- If an option has a replication price then its no-arbitrage price must be the same. Otherwise one could make arbitrage by buying or selling the option with the no-arbitrage price and then replicating the option (or the "minus option"). The price difference would be arbitrage.
- It is possible in theory that an option has a replication price but no no-arbitrage prices. It this case the market already has arbitrage.
- It is also possible to have no-arbitrage prices, but no replication prices. In this case there are many no-arbitrage prices.

Outline

Definition (Forward Integral)

Let (Π_n) be a sequence of partitions of $[0, T]$ such that

$$\mathrm{mesh}(\Pi_n) = \sup_{t_k \in \Pi_n} |t_k - t_{k-1}| \to 0, \quad n \to \infty.$$

Forward integral of f w.r.t. g on $[0, T]$ is

$$\int_0^T f(t)\,\mathrm{d}g(t) = \lim_{n\to\infty} \sum_{t_k \in \Pi_n} f(t_{k-1})\,(g(t_k) - g(t_{k-1}))\,.$$

Forward integral of f w.r.t. g on $[a, b] \subset [0, T]$ is

$$\int_a^b f(t)\,\mathrm{d}g(t) = \int_0^T f(t)\mathbf{1}_{[a,b]}(t)\,\mathrm{d}g(t).$$

Remark

- The existence (and maybe even the value) of forward integral depends on the particular choice of the sequence of partitions.
- In what follows the sequence of partitions is assumed to be refining.
- The forward integral is a pathwise (almost sure) Itô integral if the sequence of partitions is chosen properly and the integrator is a semimartingale.
- In general, there is nothing that would ensure the existence of the forward integral.
- We will see soon that if the integrator has quadratic variation then certain forward integrals will exist.

Definition (Quadratic Variation)

Let (Π_n) be a sequence of (refining) partitions of $[0, T]$ such that $\mathrm{mesh}(\Pi_n) \to 0$. Quadratic variation of f on $[0, t]$ is

$$\langle f \rangle (t) \;=\; \lim_{n\to\infty} \sum_{t_k \in \Pi_n, t_k \le t} \left(f(t_k) - f(t_{k-1})\right)^2 .$$

Remark

Quadratic covariation can be defined in the same way or by using the polarization formula

$$\langle f, g \rangle \;=\; \frac{1}{4}\left(\langle f + g \rangle - \langle f - g \rangle\right).$$

Lemma

Let f and g be continuous quadratic variation functions.

1. *For standard Brownian motion $\langle W \rangle_t = t$ a.s., if the sequence of partitions is refining.*
2. *If f is smooth then $\langle f \rangle = 0$.*
3. *If $\langle g \rangle = 0$ then $\langle f + g \rangle = \langle f \rangle$.*
4. *If f is smooth then*

$$\langle f \circ g \rangle (t) \; = \; \int_0^t f'\big(g(s)\big)^2 \, \mathrm{d}\langle g \rangle(s).$$

5. *If $\langle g \rangle = 0$ then $\langle f, g \rangle = 0$.*

THEOREM (ITÔ–FÖLLMER FORMULA)

Let $X = (X^1, \ldots, X^n)$ be a.s. continuous quadratic covariation process, and let $f : \mathbb{R}^n \to \mathbb{R}$ be smooth. Then, a.s.,

$$\mathrm{d}f(X_t) = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(X_t)\mathrm{d}X_t + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x^i \partial x^j}(X_t)\,\mathrm{d}\left\langle X^i, X^j \right\rangle_t .$$

PROOF.

Taylor is all you need! (Actually, the proof is pretty much identical to the classical proof of the Itô formula.) □

Remark

- The Itô–Föllmer formula implies that the forward integral exists and has a continuous modification.
- The Itô–Föllmer formula for quadratic variation processes is formally the same as the Itô formula for the continuous semimartingales.
- The differences between the forward world and the semimartingale worlds are:
  - The existence of the quadratic variation (limit in probability) is guaranteed for semimartingales.
  - In the forward world the integrals are always defined a.s.

Outline

Let W be the standard Brownian motion.

Definition (Black–Scholes Model)

In the Classical Black–Scholes model the (discounted) stock price process is a geometric Brownian motion

$$S_t = S_0 e^{\mu t - \frac{1}{2}\sigma^2 t + \sigma W_t},$$

or as an Itô differential

$$\mathrm{d}S_t = S_t \left(\mu \, \mathrm{d}t + \sigma \, \mathrm{d}W_t \right).$$

Definition (Completeness)

A market model is COMPLETE if all options can be replicated with a self-financing trading strategy.

Theorem

The Black–Scholes market model is complete (at least for L^2-options).

Proof.

This follows basically from the Martingale Representation Theorem combined with the Girsanov Theorem. □

Theorem

The Black–Scholes market model is free of arbitrage (for "tame" strategies).

Proof.

This follows from the fact that Itô integrals are (proper) supermartingales for "tame" integrators. □

Remark

There is arbitrage in the Black–Scholes model with e.g. doubling strategies. (The proponents of the semimartingale approach do not like to talk too much about this.)

Outline

Let $G = g(S_T)$ be a path-independent option. (We will consider path-dependent options in the final section.)

Assume that S is continuous and

$$\mathrm{d}\left\langle S\right\rangle_t \;=\; \sigma^2 S_t^2 \mathrm{d}t. \tag{2}$$

Remark

- The classical Black–Scholes model satisfies (2).
- Let

$$\mathrm{d}S_t \;=\; S_t\left(\mu\mathrm{d}t + \sigma\mathrm{d}W_t + \nu\mathrm{d}B_t^H\right),$$

  where B^H is a fractional Brownian motion with $H > 1/2$. This model also satisfies (2), but now S is no longer a semimartingale.

By Itô–Föllmer Formula, if $v(t,x)$ is the solution to
Black-Scholes BPDE

$$\frac{\partial v}{\partial t}(t,x) + \frac{\sigma^2}{2}x^2\frac{\partial^2 v}{\partial x^2}(t,x) \;=\; 0$$

with boundary condition $v(T,x) = g(x)$, then

$$V_t(\Phi) \;=\; v(t,S_t) \qquad \text{and} \qquad \Phi_t \;=\; \frac{\partial v}{\partial x}(t,S_t).$$

Indeed, now

$$\begin{aligned} V_T(\Phi) \;&=\; V_0(\Phi) + \int_0^T \Phi_t \mathrm{d}S_t \\ &\quad + \underbrace{\int_0^T \frac{\partial v}{\partial t}(t,S_t)\mathrm{d}t + \frac{\sigma^2}{2}\int_0^T \frac{\partial^2 v}{\partial x^2}(t,S_t)S_t^2\mathrm{d}t}_{=0}. \end{aligned}$$

Remark

- Note that the prices and replications of (path-independent) options were derived from the quadratic variation property. Probability does not come into it!
- It is true, by Feynman–Kac Formula, that

$$v(t,x) \;=\; \mathbf{E}\left[g\left(x\exp\left\{\sigma W_{T-t} - \frac{\sigma^2}{2}(T-t)\right\}\right)\right],$$

  where W is the Brownian motion. This does not imply that S is geometric Brownian motion!
- In the semimartingale world option price is the expectation w.r.t. the equivalent martingale measure. In the non-semimartingale world this is not completely true.

Outline

- No-arbitrage and completeness are opposite requirements: The more you want to hedge the more sophisticated trading strategies you must allow — The more sophisticated trading strategies you allow the more candidates for arbitrage you have.
- In classical semimartingale theory completeness and arbitrage are attained in the class of either integrable enough strategies or bounded from below strategies.
- In the quadratic variation world we would then like to identify a class of strategies that is
  1. Big enough to contain hedges for relevant(?) options.
  2. Small enough to exclude arbitrage opportunities.
  3. Economically meaningful.

We propose the following class of SMOOTH STRATEGIES:

$$\Phi_t = \varphi(t, S_t, g^1(t,S), \ldots g^n(t,S)),$$

where φ is smooth and g^i's are HINDSIGHT FACTORS:

1. $g^i(\cdot, S)$ is S-adapted.
2. $g^i(\cdot, S)$ is continuous bounded variation process.
3. $\left| \int_0^t f(u) \mathrm{d}g^i(u,S) - \int_0^t f(u) \mathrm{d}g^i(u,\tilde{S}) \right| \leq K \|f\|_\infty \|S - \tilde{S}\|_\infty.$

For example, running maximum, minimum, and average are hindsight factors.

Definition (Stopping-Smooth)

Trading strategy is stopping-smooth if it is of the form

$$\Phi_t = \sum_{k=1}^{n} \Phi_t^k \mathbf{1}_{(\tau_k, \tau_{k+1}]}(t),$$

where Φ^k's are smooth and the stopping times τ_k are locally continuous.

A function f is locally continuous at point x if there exist an open set U such that $x \in \bar{U}$ and $f(x_n) \to f(x)$ whenever $x_n \in U$.

Theorem

There are no arbitrage opportunities in the class of stopping-smooth strategies if S has conditional full support:

$$\mathbf{P}\left[\sup_{s\in[t,T]} |S_t - \eta| \le \varepsilon \,\middle|\, \mathcal{F}_t^S\right] > 0$$

for all positive continuous paths η s.t. $\eta(t) = S_t$.

Outline

Replication with Non-Semimartingales Revisited

Let S be continuous with the Black–Scholes quadratic variation

$$\mathrm{d}\langle S\rangle_t = \sigma^2 S_t^2 \mathrm{d}t.$$

Theorem

If an option $G = G(S)$ can be replicated in the Black–Scholes model with a smooth strategy (involving hindsight factors), then it can be replicated in the model S. Moreover, the replicating strategies are — as functionals of the observed stock prices and the quadratic variation σ^2 — the same.

Thank You for listening!
Any questions?