Pretty Predictable Models

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CMStatistics - London, 15 December 2019

Some References

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Outline

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Aim and Strategy

Let $X = (X_t)_{t\in[0,T]}$ be a stochastic process.

Our aim is to predict

$$\mathbb{P}[X_T \in B \,|\, \mathcal{F}_t],$$

where $(\mathcal{F}_t)_{t\in[0,T]}$ is the filtration generated by X.

Our strategy is to represent X via a Brownian motion W. More precisely, we go:

- $X \to G$, where G is Gaussian,
- $G \to W$, where W is Brownian motion,

and then back again.

What is essential, is that the transformations are invertible and the filtrations generated by X, G and W coincide.

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Brownian Innovation Representation

Technically, we assume

1. $X_t = g(t, G_t)$, where g is invertible and G is centered Gaussian.
2. $r(t,s) = \mathbb{E}[G_t G_s]$ is of the form

$$r(t,s) = \int_0^{t\wedge s} k(t,u)k(s,u)\,\mathrm{d}u,$$

   for some $k(t,\cdot) \in L^2([0,T])$ for all $t \in [0,T]$.
3. The equation

$$K^* f = \mathbf{1}_{[0,t)}$$

   has a solution for all $t \in [0,T]$, where K^* linearly extends the relation $K^*\mathbf{1}_{[0,t)} = k(t,\cdot)$.

Brownian Innovation Representation

The operator K^* requires some clarification, I guess.

Suppose $k(t, t-) = 0$ and $k(\cdot, t)$ is of bounded variation. Then

$$K^* f(t) = \int_t^T f(s) k(\mathrm{d}s, t)$$

Let Λ be the Hilbert space generated by the indicator functions $\mathbf{1}_{[0,t)}$, $t \in [0, T]$ and closed under the inner product

$$\langle \mathbf{1}_{[0,t)}, \mathbf{1}_{[0,s)} \rangle_\Lambda = R(t, s).$$

Then K^* is an isometry between Λ and $L^2([0, T])$.

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Transfer Principle

Let assumptions (2) and (3) hold. Let $k^{-1}(t,\cdot)$ be the solution of the equation in (3). Then $k^{-1}(t,\cdot) \in \Lambda$ and

$$W_t = \int_0^t k^{-1}(t,s)\,\mathrm{d}G_s$$

is a Brownian motion. Moreover, G can be recovered from W by

$$G_t = \int_0^t k(t,s)\,\mathrm{d}W_s.$$

These extend to transfer principle

$$\int_0^T f(t)\,\mathrm{d}G_t = \int_0^T K^* f(t)\,\mathrm{d}W_t.$$

$$\int_0^T f(t)\,\mathrm{d}W_t = \int_0^T K^{*-1} f(t)\,\mathrm{d}G_t.$$

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The Prediction Formula

A nice thing about conditional Gaussian processes are, that they are Gaussian, if the condition is Gaussian.

So, to calculate

$$\mathbb{P}[G_T \in B | \mathcal{F}_t]$$

we only need to calculate the conditional mean and conditional (co)variance.

Now,

$$G_T = \int_0^t k(T,s)\,\mathrm{d}W_s + \int_t^T k(T,s)\,\mathrm{d}W_s.$$

Consequently,

$$\begin{aligned} \hat{m}_t &= \mathbb{E}[G_T | \mathcal{F}_t] \\ &= \int_0^t k(T,s)\,\mathrm{d}W_s \end{aligned}$$

The Prediction Formula

By using the (inverse) transfer principle, we can write

$$\hat{m}_t = \int_0^t K^{*-1} k(T, \cdot)(s)\,\mathrm{d}G_s.$$

As for the (co)variance we have

$$\begin{aligned} \hat{v}(t) &= \mathbb{E}[(G_T - \hat{m}_t)^2 | \mathcal{F}_t] \\ &= \mathbb{E}\left[\left(\int_t^T k(T,s)\,\mathrm{d}W_s\right)^2 \middle| \mathcal{F}_t\right] \\ &= \int_t^T k(T,s)^2\,\mathrm{d}s. \end{aligned}$$

The Prediction Formula

Thus, we have found out that,

$$\begin{aligned}
&\mathbb{P}[X_T \in B|\mathcal{F}_t] \\
&\quad = \mathbb{P}[G_T \in g^{-1}B|\mathcal{F}_t] \\
&\quad = \frac{1}{\sqrt{2\pi \hat{v}(t)}} \int_{g^{-1}B} \exp\left\{ -\frac{1}{2}\left(\frac{y-\hat{m}_t}{\sqrt{\hat{v}(t)}}\right)^2 \right\} \mathrm{d}y.
\end{aligned}$$