

# PRICING BY HEDGING AND NO-ARBITRAGE BEYOND SEMIMARTINGALES

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Moscow, November 1, 2010

International symposium: Visions in stochastics (Leaders and their Pupils)

This talk is based on the papers

- 1 Bender, C., Sottinen, T. and Valkeila, E. (2008) Pricing by hedging and no-arbitrage beyond semimartingales. *Finance and Stochastics* **12**, 441-468.
- 2 Bender, C., Sottinen, T., and Valkeila, E. (2010) Fractional processes as models in stochastic finance. To appear in *Advanced Mathematical Methods for Finance* (Eds. G. Di Nunno and B. Oksendal) Series in Mathematical Finance, Springer.

# OUTLINE

- 1 QUADRATIC VARIATION MARKET MODELS WITH  
CONDITIONAL SMALL-BALL PROPERTY
- 2 NO-ARBITRAGE WITH ALLOWED STRATEGIES
- 3 LOCALLY CONTINUOUS STOPPING TIMES
- 4 NO-ARBITRAGE WITH STOPPING-ALLOWED  
STRATEGIES
- 5 NO-ARBITRAGE WITH SIMPLE STRATEGIES

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# QUADRATIC VARIATION MARKET MODELS WITH CONDITIONAL SMALL-BALL PROPERTY

We assume that the stock-price process  $S = (S_t)_{t \in [0, T]}$  is (almost surely) continuous, strictly positive, starts from  $s_0$ , and the information used in trading is generated by it.

So, we work in the canonical space  $\Omega = \mathcal{C}_{s_0, \sigma}[0, T]$  with  $S_t(\eta) = \eta(t)$  and

$$\mathcal{F}_t = \sigma \{ \eta(s); s \leq t \},$$

$\mathcal{F} = \mathcal{F}_T$ . (The index  $\sigma > 0$  will be explained in the next slide.)

# QUADRATIC VARIATION MARKET MODELS WITH CONDITIONAL SMALL-BALL PROPERTY

We assume that almost surely the stock-price process has the **QUADRATIC VARIATION** of the Black-Scholes model:

$$d \langle S \rangle_t = \sigma^2 S_t^2 dt.$$

We assume that the following **CONDITIONAL SMALL-BALL PROPERTY** is satisfied:

$$\mathbf{P} \left[ \sup_{t \in [\tau, T]} |S_t - \eta(t)| < \varepsilon \mid \mathcal{F}_\tau \right] > 0$$

**P**-a.s. for all paths  $\eta$ , positive  $\varepsilon$ , and stopping times  $\tau$ .

So, we have a collection of models **P** on the canonical filtered space  $\mathcal{C}_{s_0, \sigma}[0, T]$ , where **P** is restricted only by the assumptions of quadratic variation and conditional small-ball property.

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# NO-ARBITRAGE WITH ALLOWED STRATEGIES

Strategies that depend in a smooth way on time, spot, running maximum, running minimum and such one cannot make arbitrage in quadratic-variation small-ball models. These strategies are called **ALLOWED**.

The no-arbitrage result followed basically from the fact that we can write the value  $V_t(\Phi)(\eta)$  of an allowed strategy (almost surely) by using a value functional  $v(t, \eta; \varphi)$ :

$$V_t(\Phi)(\eta) = V_0(\Phi)(\eta) + v(t, \eta; \varphi) \quad \text{for } \mathbf{P}\text{-a.a. } \eta,$$

and  $v(t, \eta; \varphi)$  is continuous in  $\eta$  uniformly in  $t$ . Here  $\varphi$  is the strategy functional associated to  $\Phi$ :

$$\Phi_t(\eta) = \varphi\left(t, \eta(t), g_1(t, \eta), \dots, g_m(t, \eta)\right),$$

where  $\varphi$  is smooth and  $g_1, \dots, g_m$  are **HINDSIGHT FACTORS**.



# NO-ARBITRAGE WITH ALLOWED STRATEGIES

The allowed strategies are natural from the hedging point of view: Hedging strategies of typical options are of this type. However, from the no-arbitrage point of view the allowed strategies are not so natural: They do not include stopping times.

To include stopping times we need some technical assumptions. The key concept is **LOCAL CONTINUITY**.

While stopping times are rarely continuous, the author is not aware of any (reasonable) stopping time that is not locally continuous.

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# LOCALLY CONTINUOUS STOPPING TIMES

## DEFINITION (LOCAL CONTINUITY)

Let  $\mathcal{X}$  and  $\mathcal{Y}$  be metric spaces. A function  $f : \mathcal{X} \rightarrow \mathcal{Y}$  is **LOCALLY CONTINUOUS** if for all  $x \in \mathcal{X}$  there exists an open  $U_x \subset \mathcal{X}$  such that  $x \in \bar{U}_x$  and  $f(x_n) \rightarrow f(x)$  whenever  $x_n \rightarrow x$  in  $U_x$ .

Local continuity at  $x$  is continuity from the direction  $U_x$ . If  $x \in U_x$  then local continuity is continuity.

## EXAMPLE

An indicator  $\mathbf{1}_A : \mathbb{R} \rightarrow \mathbb{R}$

- 1 is locally continuous if  $A = \bar{G}$ ,  $G$  is open,
- 2 is not locally continuous if  $A$  has an isolated point.

# LOCALLY CONTINUOUS STOPPING TIMES

The following stopping times  $\tau : \mathcal{C}_{s_0, \sigma}[0, T] \rightarrow [0, T]$  are locally continuous.

## EXAMPLE

- 1  $\tau(\eta) = \inf\{t; \eta(t) \in F\}$ ,  $F$  is closed,
- 2  $\tau(\eta) = \inf\{t; \psi(t, \eta) \in \bar{G}\}$ ,  $\psi$  is continuous and  $G$  is open,
- 3  $\tau(\eta) = \inf\{t; (t, \eta) \in \bar{U}\}$ ,  $U$  is open.

In the case (3) above we say that  $\tau$  is **FAT**. All the stopping times in the example above are fat.

The functionals in the example above are locally continuous even if they were not stopping times.

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# NO-ARBITRAGE WITH STOPPING-ALLOWED STRATEGIES

## DEFINITION (STOPPING-ALLOWED STRATEGIES)

A trading strategy  $\Phi$  is **STOPPING-ALLOWED** if it is of the form

$$\Phi_t = \sum_{k=1}^n \Phi_t^{(k)} \mathbf{1}_{(\tau_k, \tau_{k+1}]}(t),$$

where the  $\Phi^{(k)}$ 's are allowed and  $\tau_k$ 's are locally continuous.

The definition above is understood in the conditional sense, i.e.  $\Phi^{(k)}$  may depend on  $\mathcal{F}_{\tau_k}$  and  $\tau_{k+1} \geq \tau_k$  is locally continuous in the conditioned, or quotient, space  $\mathcal{C}_{\mathcal{S}_{\tau_k}, \sigma}[\tau_k, T]$ .

# NO-ARBITRAGE WITH STOPPING-ALLOWED STRATEGIES

## THEOREM (NO-ARBITRAGE WITH STOPPING-ALLOWED STRATEGIES)

*Let  $\Phi$  be a stopping-allowed strategy. Then  $\Phi$  is not an arbitrage opportunity.*

Theorem (No-Arbitrage with Stopping-Allowed Strategies) follows by applying the conditional small-ball property  $n$  times with the following lemma:

## LEMMA (NO-ARBITRAGE WITH TAKE-THE-MONEY-AND-RUN STRATEGIES)

*Let  $\Phi$  be allowed strategy and let  $\tau$  be a locally continuous stopping time. Then  $\Phi \mathbf{1}_{[0, \tau]}$  is not an arbitrage opportunity.*

# NO-ARBITRAGE WITH STOPPING-ALLOWED STRATEGIES

## PROOF OF LEMMA (NO-ARBITRAGE WITH TAKE-THE-MONEY-AND-RUN STRATEGIES).

Let  $\Phi \mathbf{1}_{[0, \tau]}$  be a candidate for an arbitrage opportunity:  
 $V_0(\Phi \mathbf{1}_{[0, \tau]}) = 0$  and  $V_T(\Phi \mathbf{1}_{[0, \tau]}) \geq 0$   $\mathbf{P}$ -a.s., or

$$v(\tau(\eta), \eta; \varphi) \geq 0 \quad \text{for } \mathbf{P}\text{-a.a. } \eta.$$

We show that  $v(\tau(\eta), \eta; \varphi) \geq 0$  for all  $\eta$ : Suppose that  $v(\tau(\eta_0), \eta_0; \varphi) < 0$  for some  $\eta_0$ . Let  $U_{\eta_0}$  be the local continuity set of  $\tau$  at  $\eta_0$ . Since  $v(t, \cdot; \varphi)$  is continuous uniformly in  $t$  we see that  $v(\tau(\cdot), \cdot; \varphi)$  is continuous on  $U_{\eta_0}$ . So, there must be a ball  $B \subset U_{\eta_0}$  such that  $v(\tau(\eta), \eta; \varphi) < 0$  for all  $\eta \in B$ . But due to the small-ball property this means that  $\mathbf{P}[V_T(\Phi \mathbf{1}_{[0, \tau]}) < 0] > 0$ , which is a contradiction.



# NO-ARBITRAGE WITH STOPPING-ALLOWED STRATEGIES

## PROOF OF LEMMA (NO-ARBITRAGE WITH TAKE-THE-MONEY-AND-RUN STRATEGIES).

Since  $v(\tau(\eta), \eta; \varphi) \geq 0$  for all  $\eta$  we have in particular that  $V_T(\Phi \mathbf{1}_{[0, \tau]}) \geq 0$   $\tilde{\mathbf{P}}$ -a.s. ( $\tilde{\mathbf{P}}$  stands for the Black-Scholes reference model). The classical theory then tells us that  $V_T(\Phi \mathbf{1}_{[0, \tau]}) = 0$   $\tilde{\mathbf{P}}$ -a.s. Then, by using the local continuity as before, we see that  $v(\tau(\eta), \eta; \varphi) = 0$  for all  $\eta$ . But this means that  $V(\Phi \mathbf{1}_{[0, \tau]}) = 0$   $\mathbf{P}$ -a.s. So,  $\Phi \mathbf{1}_{[0, \tau]}$  is not an arbitrage opportunity.  $\square$

# NO-ARBITRAGE WITH STOPPING-ALLOWED STRATEGIES

## PROOF OF THEOREM (NO-ARBITRAGE WITH STOPPING-ALLOWED STRATEGIES).

By using the conditional small-ball property instead of an unconditional one we see that Lemma (No-Arbitrage with Take-the-Money-and-Run Strategies) can be strengthened to:

$$\Phi^{(k)} \mathbf{1}_{(\tau_k, \tau_{k+1}]}$$

is not an arbitrage opportunity. Here the allowed strategy  $\Phi^{(k)}$  may depend additionally on  $\mathcal{F}_{\tau_k}$ , and  $\tau_{k+1}$  is locally continuous on the quotient, or conditioned, space  $\mathcal{C}_{S_{\tau_k}, \sigma}[\tau_k, T]$ .

But this means that the stopping-allowed strategy  $\Phi$  does not generate arbitrage on any of the stochastic intervals  $(\tau_k, \tau_{k+1}]$ . Hence, it cannot generate arbitrage on the interval  $[0, T]$ .  $\square$

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# NO-ARBITRAGE WITH SIMPLE STRATEGIES

We have shown that combinations of allowed strategies are free of arbitrage if the switching between the strategies is done with locally continuous stopping times.

Following an allowed strategy means continuous trading. In practise continuous trading is impossible: Trading strategy is constant between switching.

If we assume that the trading strategy is constant between the switching stopping times we can weaken the local continuity assumption.

One way to weaken the assumption is to ask only local lower semi-continuity instead of local “full” continuity.

# NO-ARBITRAGE WITH SIMPLE STRATEGIES

## DEFINITION (LOCAL LOWER SEMI-CONTINUITY)

Let  $\mathcal{X}$  be a metric space and let  $\mathcal{Y}$  be an ordered complete metric space. A function  $f : \mathcal{X} \rightarrow \mathcal{Y}$  is **LOCALLY LOWER SEMI-CONTINUOUS** if for all  $x \in \mathcal{X}$  there exists an open  $U_x \subset \mathcal{X}$  such that  $x \in \bar{U}_x$  and  $\liminf f(x_n) \geq f(x)$  whenever  $x_n \rightarrow x$  in  $U_x$ .

## EXAMPLE

An indicator  $\mathbf{1}_A : \mathcal{X} \rightarrow \mathbb{R}$  is locally lower semi-continuous if for all  $x \in A$  and  $\varepsilon > 0$  there exists a ball  $B \subset A$  such that  $\text{dist}(x, B) < \varepsilon$ .

# NO-ARBITRAGE WITH SIMPLE STRATEGIES

## DEFINITION (SIMPLE STRATEGY)

A trading  $\Phi$  strategy is **SIMPLE** if it is of the form

$$\Phi_t = \sum_{k=1}^n \xi_k \mathbf{1}_{(\tau_k, \tau_{k+1}]},$$

where  $\tau_k$  is locally lower semi-continuous stopping times (relative to  $\tau_{k-1}$ ) and  $\xi_k$ 's are  $\mathcal{F}_{\tau_k}$  measurable.

## THEOREM (NO-ARBITRAGE WITH SIMPLE STRATEGIES)

*Let  $\Phi$  be a simple strategy. Then  $\Phi$  is not an arbitrage opportunity.*

# NO-ARBITRAGE WITH SIMPLE STRATEGIES

Before going to the proof of Theorem (No-Arbitrage with Simple Strategies) let us note that the theorem is true even without the assumption on the quadratic variation. In contrast, Theorem (No-Arbitrage with Stopping-Allowed Strategies) fails to be true if the quadratic variation vanishes.

Because of the “time-linearity” of the arbitrage and conditional small-ball property it is enough to show the following:

## LEMMA (UP’N’DOWN)

*Let  $\tau$  be locally lower semi-continuous stopping time. Then*

$$\mathbf{P}[S_\tau > s_0] > 0 \quad \text{and} \quad \mathbf{P}[S_\tau < s_0] > 0.$$

# NO-ARBITRAGE WITH SIMPLE STRATEGIES

## PROOF OF LEMMA (UP'N'DOWN).

We show that  $\mathbf{P}[S_\tau > s_0] > 0$ ; the case  $\mathbf{P}[S_\tau < s_0] > 0$  is symmetric.

We show that the set  $\{S_\tau > s_0\} = \{\eta; \eta(\tau(\eta)) > s_0\}$  contains a ball. Then the claim will follow from the small-ball property.

Fix an increasing and concave path  $\eta_0$  with  $\eta_0(0) = s_0$  and a local lower semi-continuity set  $U_{\eta_0}$  of  $\tau$  at  $\eta_0$ .

Since  $\tau$  is lower semi-continuous on  $U_{\eta_0}$  we can find such an  $\varepsilon < 1/2 (\eta_0(\tau(\eta_0)) - s_0)$  that  $\tau(\eta) \geq 1/2 \tau(\eta_0)$  whenever  $\eta \in B$ , where  $B$  is some ball contained in  $B_{\eta_0}(\varepsilon) \cap U_{\eta_0}$ .



# NO-ARBITRAGE WITH SIMPLE STRATEGIES

## PROOF OF LEMMA (UP'N'DOWN).

Since  $\eta_0$  is increasing and concave

$$\begin{aligned}\eta(\tau(\eta)) &> \eta_0(\tau(\eta)) - 1/2 (\eta_0(\tau(\eta_0)) - s_0) \\ &\geq \eta_0(1/2 \tau(\eta_0)) - 1/2 \eta_0(\tau(\eta_0)) + 1/2 s_0 \\ &\geq 1/2 \eta_0(0) + 1/2 s_0 = s_0.\end{aligned}$$

So, the ball  $B$  is contained in the set  $\{S_\tau > s_0\}$ , which implies that  $\mathbf{P}[S_\tau > s_0] > 0$ .  $\square$

# NO-ARBITRAGE WITH SIMPLE STRATEGIES

## REMARK ( $\varepsilon$ -DELAY)

The Lemma (Up'n'Down) is true with local lower semi-continuity replaced by a weaker assumption of  $\varepsilon$ -DELAY:

For all  $\eta_0$  there are positive  $\varepsilon = \varepsilon(\eta_0)$  and  $\delta = \delta(\eta_0)$  such that

$$\tau(\eta) \geq \varepsilon \quad \text{when } \eta \in B_{\eta_0}(\delta).$$

- The End -