

# ON CONDITIONAL FULL SUPPORT WITH APPLICATIONS TO MATHEMATICAL FINANCE

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# ABSTRACT

Roughly speaking the law of a stochastic process has **CONDITIONAL FULL SUPPORT** (CFS) if, after following the path of a process upto any stopping time, all future paths are still possible.

We give criteria for checking when the law of a stochastic process has CFS and discuss the role of the CFS in non-semimartingale mathematical finance.

# OUTLINE

- 1 DEFINITION AND MOTIVATION OF CONDITIONAL FULL SUPPORT
- 2 ELEMENTARY PROPERTIES OF CONDITIONAL FULL SUPPORT
- 3 APPLICATION TO OPTION-PRICING (NO-ARBITRAGE)
- 4 REFERENCES

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# DEFINITION AND MOTIVATION OF CONDITIONAL FULL SUPPORT

## DEFINITION

### DEFINITION (CONDITIONAL FULL SUPPORT)

A continuous  $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$ -adapted process  $X = (X_t)_{t \in [0, T]}$  has  **$\mathbb{F}$ -CONDITIONAL FULL SUPPORT** ( $\mathbb{F}$ -CFS) if

$$\mathbf{P} \left[ \sup_{t \in [\tau, T]} |X_t - \eta(t)| < \varepsilon \mid \mathcal{F}_\tau \right] > 0$$

$\mathbf{P}$ -a.s. for all paths  $\eta \in C_{X_\tau}[\tau, T]$ ,  $\varepsilon > 0$ , and stopping times  $\tau \in \mathbb{F}$ -stopping times  $\tau$ .

### REMARK

By Guasoni, Rásonyi, and Schachermayer (2008) it is enough to consider deterministic times  $\tau \in [0, T]$  in the definition above.

# DEFINITION AND MOTIVATION OF CONDITIONAL FULL SUPPORT

## MOTIVATION

- Guasoni, Rásonyi and Schachermayer (2008) showed that CFS implies No-Arbitrage, when proportional transaction costs are imposed.
- CFS and existence quadratic variation imply No-Arbitrage in the (frictionless) pricing framework of Bender, S. and Valkeila (2008).
- Arguably, CFS is an interesting fundamental property from a purely mathematical point of view.

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# ELEMENTARY PROPERTIES OF CONDITIONAL FULL SUPPORT

LEMMA (GASBARRA, S., VAN ZANTEN, PAKKANEN)

- 1 If  $\mathbb{G} \subset \mathbb{F}$  and  $X$  has  $\mathbb{F}$ -CFS then  $X$  has  $\mathbb{G}$ -CFS.
- 2 If  $X$  has  $\mathbb{F}^X$ -CFS and  $X$  and  $Y$  are equivalent in law then  $Y$  has  $\mathbb{F}^Y$ -CFS.
- 3  $X$  has  $\mathbb{F}$ -CFS if and only if  $X$  has  $\bar{\mathbb{F}}$ -CFS, where  $\bar{\mathbb{F}}$  is the usual augmentation of  $\mathbb{F}$ .
- 4 If  $Y$  and  $X$  are independent, and  $X$  has  $\mathbb{F}^X$ -CFS, then  $X + Y$  has  $\mathbb{F}^{X+Y}$ -CFS.

# CONDITIONAL FULL SUPPORT FOR GAUSSIAN PROCESSES

## LEMMA (CHERNY)

*Let  $X$  be moving average process*

$$X_t = \int_{-\infty}^t \left( f(t-s) - f(-s) \right) dW_s,$$

*where  $W$  is a standard Brownian motion and  $f \in L^2(\mathbb{R}_+, \mathbb{R})$  is non-vanishing. Then  $X$  has  $\mathbb{F}^X$ -CFS.*

# CONDITIONAL FULL SUPPORT FOR GAUSSIAN PROCESSES

LEMMA (GASBARRA, S., VAN ZANTEN)

Let  $X$  be a stationary-increment Gaussian with spectral measure  $\mu$ .

1 If

$$\int_{\lambda_0}^{\infty} \frac{\log \hat{\mu}(\lambda)}{\lambda^2} d\lambda > -\infty,$$

then  $X$  has  $\mathbb{F}^X$ -CFS.

2 If

$$\int_{\lambda_0}^{\infty} e^{a\lambda} d\mu(\lambda) < \infty,$$

then  $X$  does not have  $\mathbb{F}^X$ -CFS.

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# APPLICATION TO OPTION-PRICING (NO-ARBITRAGE)

## LOCAL CONTINUITY

### DEFINITION (LOCAL CONTINUITY)

Let  $\mathcal{X}$  and  $\mathcal{Y}$  be metric spaces. A function  $f : \mathcal{X} \rightarrow \mathcal{Y}$  is **LOCALLY CONTINUOUS** (LC) if for all  $x \in \mathcal{X}$  there exists an open  $U_x \subset \mathcal{X}$  such that  $x \in \bar{U}_x$  and  $f(x_n) \rightarrow f(x)$  whenever  $x_n \rightarrow x$  in  $U_x$ .

### REMARK

- LC at  $x$  is continuity from the “direction”  $U_x$ . However, LC is not directional continuity in the classical sense. If  $x \in U_x$  then LC is classical continuity.



$$\tau(\eta) = \inf\{t; \eta(t) \in F\}, \quad F \text{ closed}$$

is LC.

# APPLICATION TO OPTION-PRICING (NO-ARBITRAGE) PRICING MODELS

- The **ASSET-PRICE**  $S$  is continuous Quadratic Variation (QV) process with

$$d\langle S \rangle_t = \sigma^2 S_t^2 dt$$

that has  $\mathbb{F}^S$ -CFS (on  $C_{S_0,+}[0, T]$ ).

- For trading **TRADING STRATEGIES** we assume that they are admissible in the classical sense, and have the form

$$\Phi_t = \sum_{k=1}^n \Phi_t^{(k)} \mathbf{1}_{(\tau_{k-1}, \tau_k]}(t)$$

where  $\tau_k$ 's are Locally Continuous (LC) stopping times and

$$\Phi_t^k = \varphi_k \left( t, S_t, \max_{s \leq t} S_s, \min_{s \leq t} S_s, \int_0^t S_s ds \right), \varphi_k \in C^1.$$

We call these strategies **ALLOWED**.

# APPLICATION TO OPTION-PRICING (NO-ARBITRAGE)

## NO-ARBITRAGE WITH ALLOWED STRATEGIES

### LEMMA (NO-ARBITRAGE WITH TAKE-THE-MONEY-AND-RUN)

*Let  $\Phi$  be allowed strategy and let  $\tau$  be a locally continuous stopping time. Then  $\Phi \mathbf{1}_{[0, \tau]}$  is not an arbitrage opportunity.*

### PROOF.

Let  $\Phi \mathbf{1}_{[0, \tau]}$  be a candidate for arbitrage:  $V_0(\Phi \mathbf{1}_{[0, \tau]}) = 0$  and  $V_T(\Phi \mathbf{1}_{[0, \tau]}) \geq 0$   $\mathbf{P}$ -a.s., or  $v(\tau(\eta), \eta; \varphi) \geq 0$   $\mathbf{P}$ -a.a.  $\eta$ . We show that  $v(\tau(\eta), \eta; \varphi) \geq 0$  for all  $\eta$ : Suppose that  $v(\tau(\eta_0), \eta_0; \varphi) < 0$  for some  $\eta_0$ . Let  $U_{\eta_0}$  be a LC set of  $\tau$  at  $\eta_0$ . Since  $v(t, \cdot; \varphi)$  is continuous,  $v(\tau(\cdot), \cdot; \varphi)$  is continuous on  $U_{\eta_0}$ . So, there is a ball  $B \subset U_{\eta_0}$  such that  $v(\tau(\eta), \eta; \varphi) < 0$  for all  $\eta \in B$ . By (C)FS this means that  $\mathbf{P}[V_T(\Phi \mathbf{1}_{[0, \tau]}) < 0] > 0$ , a contradiction.

# APPLICATION TO OPTION-PRICING (NO-ARBITRAGE)

## NO-ARBITRAGE WITH ALLOWED STRATEGIES

### PROOF OF LEMMA (NO-ARBITRAGE WITH TAKE-THE-MONEY-AND-RUN).

Since  $v(\tau(\eta), \eta; \varphi) \geq 0$  for all  $\eta$  we have  $V_T(\Phi \mathbf{1}_{[0, \tau]}) \geq 0$   $\tilde{\mathbf{P}}$ -a.s. ( $\tilde{\mathbf{P}}$  stands for the Black-Scholes reference model). By the classical theory  $V_T(\Phi \mathbf{1}_{[0, \tau]}) = 0$   $\tilde{\mathbf{P}}$ -a.s. Then, by using LC  $v(\tau(\eta), \eta; \varphi) = 0$  for all  $\eta$ . But this means that  $V(\Phi \mathbf{1}_{[0, \tau]}) = 0$   $\mathbf{P}$ -a.s. So,  $\Phi \mathbf{1}_{[0, \tau]}$  is not an arbitrage opportunity.  $\square$

### THEOREM (NO-ARBITRAGE WITH STOPPING-ALLOWED STRATEGIES)

*Let  $\Phi$  be a stopping-allowed strategy. Then  $\Phi$  is not an arbitrage opportunity.*

# APPLICATION TO OPTION-PRICING (NO-ARBITRAGE)

## NO-ARBITRAGE WITH ALLOWED STRATEGIES

### PROOF.

By CFS Lemma (No-Arbitrage with Take-the-Money-and-Run) can be strengthened to:

$$\Phi^{(k)} \mathbf{1}_{(\tau_k, \tau_{k+1}]}$$

is not an arbitrage opportunity. Here the allowed strategy  $\Phi^{(k)}$  may depend additionally on  $\mathbb{F}_{\tau_k}$ , and  $\tau_{k+1}$  is locally continuous on the quotient, or conditioned, space  $C_{S_{\tau_k}, +}[\tau_k, T]$ .

But this means that the stopping-allowed strategy  $\Phi$  does not generate arbitrage on any of the stochastic intervals  $(\tau_k, \tau_{k+1}]$ . Hence, it cannot generate arbitrage on the interval  $[0, T]$ .  $\square$

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- C. Bender, T. Sottinen, E. Valkeila (2008) Pricing by hedging and no-arbitrage beyond semimartingales, *Finance Stoch.* **12** (4), 441–468.
- A. Cherny (2008) Brownian moving averages have conditional full support, *Ann. Appl. Probab.* **18** (5) 1825–1830.
- D. Gasbarra, T. Sottinen, H. van Zanten (2008) Conditional full support of Gaussian processes with stationary increments, *Preprint 487*, Department of Mathematics and Statistics, University of Helsinki.
- P. Guasoni, M. Rásonyi, W. Schachermayer (2008) Consistent price systems and face-lifting pricing under transaction costs, *Ann. Appl. Probab.* **18** (2) 491–520.
- M. Pakkanen (2008) Stochastic integrals and conditional full support. *Preprint*.