On Conditional Full Support with Applications to Mathematical Finance

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Abstract

Roughly speaking the law of a stochastic process has Conditional Full Support (CFS) if, after following the path of a process upto any stopping time, all future paths are still possible.

We give criteria for checking when the law of a stochastic process has CFS and discuss the role of the CFS in non-semimartingale mathematical finance.

Outline

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Definition (Conditional Full Support)

A continuous $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$-adapted process $X = (X_t)_{t\in[0,T]}$ has $\mathbb{F}$-Conditional Full Support ($\mathbb{F}$-CFS) if

$$\mathbf{P}\left[\sup_{t\in[\tau,T]} |X_t - \eta(t)| < \varepsilon \,\middle|\, \mathcal{F}_\tau\right] > 0$$

P-a.s. for all paths $\eta \in C_{X_\tau}[\tau, T]$, $\varepsilon > 0$, and stopping times $\tau \in \mathbb{F}$-stopping times τ.

Remark

By Guasoni, Rásonyi, and Schachermayer (2008) it is enough to consider deterministic times $\tau \in [0, T]$ in the definition above.

- Guasoni, Rásonyi and Schachermayer (2008) showed that CFS implies No-Arbitrage, when proportional transaction costs are imposed.
- CFS and existence quadratic variation imply No-Arbitrage in the (frictionless) pricing framework of Bender, S. and Valkeila (2008).
- Arguably, CFS is an interesting fundamental property from a purely mathematical point of view.

Outline

Elementary Properties of Conditional Full Support

Lemma (Gasbarra, S., Van Zanten, Pakkanen)

1. *If $\mathbb{G} \subset \mathbb{F}$ and X has $\mathbb{F}$-CFS then X has $\mathbb{G}$-CFS.*
2. *If X has $\mathbb{F}^X$-CFS and X and Y are equivalent in law then Y has $\mathbb{F}^Y$-CFS.*
3. *X has $\mathbb{F}$-CFS if and only if X has $\bar{\mathbb{F}}$-CFS, where $\bar{\mathbb{F}}$ is the usual augmentation of $\mathbb{F}$.*
4. *If Y and X are independent, and X has $\mathbb{F}^X$-CFS, then $X + Y$ has $\mathbb{F}^{X+Y}$-CFS.*

Conditional Full Support for Gaussian Processes

Lemma (Cherny)

Let X be moving average process

$$X_t = \int_{-\infty}^{t} \Big(f(t-s) - f(-s) \Big) \, \mathrm{d}W_s,$$

where W is a standard Brownian motion and $f \in L^2(\mathbb{R}_+, \mathbb{R})$ is non-vanishing. Then X has $\mathbb{F}^X$-CFS.

Conditional Full Support for Gaussian Processes

Lemma (Gasbarra, S., Van Zanten)

Let X be a stationary-increment Gaussian with spectral measure μ.

1. *If*

$$\int_{\lambda_0}^{\infty} \frac{\log \dot{\mu}(\lambda)}{\lambda^2}\, \mathrm{d}\lambda > -\infty,$$

   then X has $\mathbb{F}^X$-CFS.

2. *If*

$$\int_{\lambda_0}^{\infty} e^{a\lambda}\, \mathrm{d}\mu(\lambda) < \infty,$$

   then X does not have $\mathbb{F}^X$-CFS.

Outline

Definition (Local Continuity)

Let $\mathcal{X}$ and $\mathcal{Y}$ be metric spaces. A function $f : \mathcal{X} \to \mathcal{Y}$ is Locally Continuous (LC) if for all $x \in \mathcal{X}$ there exists an open $U_x \subset \mathcal{X}$ such that $x \in \bar{U}_x$ and $f(x_n) \to f(x)$ whenever $x_n \to x$ in U_x.

Remark

- LC at x is continuity from the "direction" U_x. However, LC is not directional continuity in the classical sense. If $x \in U_x$ then LC is classical continuity.
- $$\tau(\eta) = \inf\{t\,;\, \eta(t) \in F\}, \quad F \text{ closed}$$

  is LC.

- The ASSET-PRICE S is continuous Quadratic Variation (QV) process with
$$\mathrm{d}\langle S\rangle_t = \sigma^2 S_t^2 \mathrm{d}t$$
that has $\mathbb{F}^S$-CFS (on $C_{S_0,+}[0,T]$).
- For trading TRADING STRATEGIES we assume that they are admissible in the classical sense, and have the form
$$\Phi_t = \sum_{k=1}^{n} \Phi_t^{(k)} \mathbf{1}_{(\tau_{k-1},\tau_k]}(t)$$
where τ_k's are Locally Continuous (LC) stopping times and
$$\Phi_t^k = \varphi_k \left(t, S_t, \max_{s\leq t} S_s, \min_{s\leq t} S_s, \int_0^t S_s \mathrm{d}s \right), \varphi_k \in C^1.$$
We call these strategies ALLOWED.

Lemma (No-Arbitrage with Take-the-Money-and-Run)

Let Φ be allowed strategy and let τ be a locally continuous stopping time. Then $\Phi\mathbf{1}_{[0,\tau]}$ is not an arbitrage opportunity.

Proof.

Let $\Phi\mathbf{1}_{[0,\tau]}$ be a candidate for arbitrage: $V_0(\Phi\mathbf{1}_{[0,\tau]}) = 0$ and $V_T(\Phi\mathbf{1}_{[0,\tau]}) \geq 0$ **P**-a.s., or $v(\tau(\eta), \eta; \varphi) \geq 0$ **P**-a.a. η. We show that $v(\tau(\eta), \eta; \varphi) \geq 0$ for all η: Suppose that $v(\tau(\eta_0), \eta_0; \varphi) < 0$ for some η_0. Let U_{η_0} be a LC set of τ at η_0. Since $v(t, \cdot; \varphi)$ is continuous, $v(\tau(\cdot), \cdot; \varphi)$ is continuous on U_{η_0}. So, there is a ball $B \subset U_{\eta_0}$ such that $v(\tau(\eta), \eta; \varphi) < 0$ for all $\eta \in B$. By (C)FS this means that $\mathbf{P}[V_T(\Phi\mathbf{1}_{[0,\tau]}) < 0] > 0$, a contradiction.

Proof of Lemma (No-Arbitrage with Take-the-Money-and-Run).

Since $v(\tau(\eta), \eta; \varphi) \geq 0$ for all η we have $V_T(\Phi \mathbf{1}_{[0,\tau]}) \geq 0$ $\tilde{\mathbf{P}}$-a.s. ($\tilde{\mathbf{P}}$ stands for the Black-Scholes reference model). By the classical theory $V_T(\Phi \mathbf{1}_{[0,\tau]}) = 0$ $\tilde{\mathbf{P}}$-a.s. Then, by using LC $v(\tau(\eta), \eta; \varphi) = 0$ for all η. But this means that $V(\Phi \mathbf{1}_{[0,\tau]}) = 0$ $\mathbf{P}$-a.s. So, $\Phi \mathbf{1}_{[0,\tau]}$ is not an arbitrage opportunity. □

Theorem (No-Arbitrage with Stopping-Allowed Strategies)

Let Φ be a stopping-allowed strategy. Then Φ is not an arbitrage opportunity.

Proof.

By CFS Lemma (No-Arbitrage with Take-the-Money-and-Run) can be strengthened to:

$$\Phi^{(k)}\mathbf{1}_{(\tau_k,\tau_{k+1}]}$$

is not an arbitrage opportunity. Here the allowed strategy $\Phi^{(k)}$ may depend additionally on $\mathbb{F}_{\tau_k}$, and τ_{k+1} is locally continuous on the quotient, or conditioned, space $C_{S_{\tau_k},+}[\tau_k, T]$.

But this means that the stopping-allowed strategy Φ does not generate arbitrage on any of the stochastic intervals $(\tau_k, \tau_{k+1}]$. Hence, it cannot generate arbitrage on the interval $[0, T]$. □

Outline

References

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