Integration-by-Parts Characterizations of Gaussian Processes

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Workshop on Stochastics, Memory and Roughness 2024, Oslo, 18 Jan 2024

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  Integration-by-Parts Characterizations of Gaussian Processes
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  doi:10.1007/s13348-019-00278-x

Abstract

The Stein's lemma characterizes the Gaussian distribution via an integration-by-parts formula.

We show that a similar integration-by-parts formula characterizes a wide class of Gaussian processes, the so-called Gaussian Fredholm processes.

Outline

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Stein's (Multivariate) Lemma

Stein's Lemma, a.k.a. the Integration-by-Parts Characterization, states that a random variable X is standard normal if and only if

$$\mathbf{E}\left[f'(X)\right] = \mathbf{E}\left[Xf(X)\right]$$

for all smooth and bounded enough $f\colon \mathbb{R} \to \mathbb{R}$.

Multivariate Stein's Lemma states that $X = (X_1, \ldots, X_d)$ is centered Gaussian with covariance R if and only if

$$\mathbf{E}\left[\sum_{i=1}^{d} X_i \frac{\partial}{\partial x_i} f(X)\right] = \mathbf{E}\left[\sum_{i=1}^{d}\sum_{j=1}^{d} R_{ij} \frac{\partial^2}{\partial x_i \partial x_j} f(X)\right]$$

for all smooth and bounded enough $f\colon \mathbb{R}^d \to \mathbb{R}$

Stein's (Multivariate) Lemma

Let $X = (X_t)_{t\in[0,1]}$ be a centered process with covariance R. The Multivariate Stein's Lemma suggests us to guess (wrongly!) that X is Gaussian if and only if

$$\mathbf{E}\left[\int_0^1 X_t D_t f(X)\,\mathrm{d}t\right] = \mathbf{E}\left[\int_0^1\int_0^1 R(t,s) D^2_{t,s} f(X)\,\mathrm{d}s\mathrm{d}t\right],$$

where D is some kind of Malliavin Derivative.

Outline

1. Stein's (Multivariate) Lemma

2. Fredholm Representation

3. Pathwise Malliavin Differentiation

4. Strong Form Integration-by-Parts Characterization

5. Weak Form Integration-by-Parts Characterization

Theorem (Fredholm Representation)

Let $X = (X_t)_{t\in[0,1]}$ be a separable centered Gaussian process. Then there exists a kernel $K \in L^2 \times L^2 = L^2([0,1]^2)$ and a Brownian motion $W = (W_t)_{t\geq 0}$ such that

$$X_t = \int_0^1 K(t,s)\,\mathrm{d}W_s$$

if and only if the covariance R of X satisfies the trace condition

$$\int_0^T R(t,t)\,\mathrm{d}t < \infty.$$

Outline

Pathwise Malliavin Differentiation

Let $\mathcal{C}_p^\infty(\mathbb{R}^n)$ denote the space of all polynomially bounded functions with polynomially bounded partial derivatives of all orders. Consider functionals $f\colon L^2 \to \mathbb{C}$ of the form

$$f(x) = g\left(z_1, \dots, z_n\right),$$

where $n \in \mathbb{N}$ and $g \in \mathcal{C}_p^\infty(\mathbb{R}^n)$, and

$$z_k = \int_0^1 e_k(t)\, \mathrm{d}x(t)$$

for some elementary functions $e_k \in \mathcal{E}$. For such f we write $f \in \mathcal{S}$. We call the elements of class $\mathcal{S}$ the SMOOTH functionals. The PATHWISE MALLIAVIN DERIVATIVE of such $f \in \mathcal{S}$ is

$$D_t f(x) = \sum_{k=1}^{n} \frac{\partial}{\partial z_k} g(z_1, \dots, z_n)\, e_k(t).$$

Pathwise Malliavin Differentiation

More generally, by iteration for every $m \in \mathbb{N}$, the pathwise Malliavin derivative of order m is defined as follows: for every $t_1, \ldots, t_m \in [0,1]$,

$$D^m_{t_m,\ldots,t_1} f(x) = \sum_{1 \le k_1,\ldots,k_m \le n} \frac{\partial^m}{\partial z_{k_1} \cdots \partial z_{k_m}} g(z_{k_1}, \ldots, z_{k_n}) \left(e_{k_1} \otimes \cdots \otimes e_{k_m}\right)(t_1, \ldots, t_m).$$

Remark

Let $f \in \mathcal{S}$ and $y \in L^2$. Then

$$\langle \nabla f(x), Iy \rangle_{L^2} = \langle Df(x), y \rangle_{L^2}$$

Outline

Strong Form Integration-by-Parts Characterization

Let K^* extend linearly the relation $K^*\mathbf{1}_t(s) = K(t,s)$, where $\mathbf{1}_t = \mathbf{1}_{[0,t)}$.

Theorem

Let $K \in L^2 \times L^2$. The co-ordinate process $X\colon \Omega \to L^2$ is centered Gaussian with Fredholm kernel K if and only if

$$\mathbf{E}\left[X_t D_t f(X)\right] = \mathbf{E}\left[\int_0^1 K(t,s) K^*\left[D^2_{t,\cdot} f(X)\right](s)\,\mathrm{d}s\right]$$

for all $t \in [0,1]$ and $f \in \mathcal{S}$.

Strong IBP, Proof of If Part

Suppose the co-ordinate process $X\colon \Omega \to L^2$ satisfies the Strong IBP Formula. We begin by considering the covariance function of X, which will justify the use of the Fubini theorem later and make a tedious variance calculations unnecessary. For this, take $f(X) = \frac{1}{2}X_u^2$ for some $u \in [0,1]$. Then $f \in \mathcal{S}$. We have $D_t f(X) = X_u \mathbf{1}_u(t)$ and $D^2_{t,s} f(X) = \mathbf{1}_u(s)\mathbf{1}_u(t)$. Consequently,

$$
\begin{aligned}
\mathbf{E}\left[X_t X_u\right]\mathbf{1}_u(t) &= \mathbf{E}\left[\int_0^1 K(t,s)K^*\left[\mathbf{1}_u\right](s)\,\mathrm{d}s\right]\mathbf{1}_u(t) \\
&= \int_0^1 K(t,s)K(u,s)\,\mathrm{d}s\,\mathbf{1}_u(t) \\
&= R(t,u)\mathbf{1}_u(t).
\end{aligned}
$$

This shows that X has the covariance function R given by the Fredholm kernel K.

Strong IBP, Proof of If Part

In particular, we have

$$\mathbf{E}\left[X_t^2\right] = \int_0^1 K(t,s)^2 \, \mathrm{d}s,$$

and since $K \in L^2 \times L^2$, we have $\int_0^1 \mathbf{E}\left[X_t^2\right] \, \mathrm{d}t < \infty$ which justifies the use of the Fubini theorem in the rest of the proof. Next we are going to show that any finite linear combination

$$Z = \sum_{k=1}^{n} a_k \left(X_{t_k} - X_{t_{k-1}}\right) = \int_0^1 e(t) \, \mathrm{d}X_t$$

with $e = \sum_k^n a_k \mathbf{1}_{(t_{k-1}, t_k]} \in \mathcal{E}$ is a Gaussian random variable. Now, note that for every θ the complex-valued exponential functional $e^{\mathrm{i}\theta Z} = \cos(\theta Z) + \mathrm{i}\sin(\theta Z)$ belongs to $\mathcal{S}$, meaning that the real and imaginary parts both belong to $\mathcal{S}$.

Strong IBP, Proof of If Part

Let φ be the characteristic function of Z. Then

$$\begin{aligned} D_t \mathrm{e}^{\mathrm{i}\theta Z} &= \mathrm{i}\theta e(t)\, \mathrm{e}^{\mathrm{i}\theta Z}, \\ D^2_{t,s} \mathrm{e}^{\mathrm{i}\theta Z} &= -\theta^2 e(t) e(s)\, \mathrm{e}^{\mathrm{i}\theta Z}. \end{aligned}$$

Hence $\mathbf{E}\left[X_t D_t \mathrm{e}^{\mathrm{i}\theta Z}\right] = \mathrm{i}\theta e(t) \mathbf{E}\left[X_t \mathrm{e}^{\mathrm{i}\theta Z}\right]$. Also, by a direct application of Fubini theorem

$$\begin{aligned} &\mathbf{E}\left[\int_0^1 K(t,s) K^*\left[D^2_{t,\cdot}\, \mathrm{e}^{\mathrm{i}\theta Z}\right](s)\, \mathrm{d}s\right] \\ &= -\mathbf{E}\left[\int_0^1 K(t,s) K^*\left[\theta^2 e(t) e(\cdot) \mathrm{e}^{\mathrm{i}\theta Z}\right](s)\, \mathrm{d}s\right] \\ &= -\theta^2 e(t) \mathbf{E}\left[\int_0^1 K(t,s) K^*\left[e(\cdot) \mathrm{e}^{\mathrm{i}\theta Z}\right](s)\, \mathrm{d}s\right] \\ &= -\theta^2 e(t) \int_0^1 K(t,s) e^*(s)\, \mathrm{d}s\ \mathbf{E}\left[\mathrm{e}^{\mathrm{i}\theta Z}\right], \end{aligned}$$

Strong IBP, Proof of If Part

where we have denoted $e^* = K^* e$. Consequently, the Strong IBP Formula yields

$$\mathrm{i}\,\mathbf{E}\left[X_t \mathrm{e}^{i\theta Z}\right] = -\theta \int_0^1 K(t,s)e^*(s)\,\mathrm{d}s\ \varphi(\theta).$$

By Fubini theorem justified by the covariance computation, we also have

$$\varphi'(\theta) = \mathbf{E}\left[\mathrm{i}Z\,\mathrm{e}^{\mathrm{i}\theta Z}\right].$$

Thus we obtain that $\varphi'(\theta) = -c\theta\,\varphi(\theta)$, where we have denoted

$$c = \int_0^1 \left(\sum_{k=1}^n a_k \left(K(t_k,s) - K(t_{k-1},s)\right) e^*(s)\right)\,\mathrm{d}s < \infty.$$

This implies that $\varphi(\theta) = \mathrm{e}^{-\frac{1}{2}c\theta^2}$, and since φ is a characteristic function, $c > 0$. Consequently, Z is a centered Gaussian random variable with variance c.

Strong IBP, Proof of Only If Part

Since the co-ordinate process $X\colon \Omega \to L^2$ is Gaussian, we have the full power of Malliavin calculus at our disposal.

In particular, we can use Malliavin IBP Formula

$$\mathbf{E}[FG] = \mathbf{E}[F]\mathbf{E}[G] + \mathbf{E}\left[\langle DF, -D\mathrm{L}^{-1}G\rangle_{\mathcal{H}}\right].$$

with $F = D_t f(X)$ and $G = X_t$. Since $\mathbf{E}[X_t] = 0$, we obtain

$$\mathbf{E}\left[X_t D_t f(X)\right] = \mathbf{E}\left[\langle D^2_{t,\cdot} f(X), -D\mathrm{L}^{-1}X_t\rangle_{\mathcal{I}}\right]$$

But $-D\mathrm{L}^{-1}X_t = \mathbf{1}_t$ and K^* is an isometry between $\mathcal{I}$ and L^2. Therefore, by noticing that $K^*\mathbf{1}_t(s) = K(t,s)$, we obtain

Strong IBP, Proof of Only If Part

$$
\begin{aligned}
\mathbf{E}\left[X_t D_t f(X)\right] &= \mathbf{E}\left[\left\langle D^2_{t,\cdot} f(X), \mathbf{1}_t \right\rangle_{\mathcal{I}}\right] \\
&= \mathbf{E}\left[\left\langle K^* D^2_{t,\cdot} f(X), K^* \mathbf{1}_t \right\rangle_{L^2}\right] \\
&= \mathbf{E}\left[\int_0^1 K^* \left[D^2_{t,\cdot} f(X)\right](s) K^* \mathbf{1}_t(s)\, \mathrm{d}s\right] \\
&= \mathbf{E}\left[\int_0^1 K^* \left[D^2_{t,\cdot} f(X)\right](s) K(t,s)\, \mathrm{d}s\right]
\end{aligned}
$$

showing the claim.

Outline

Weak Form Integration-by-Parts Characterization

By using Fubini to the Strong IBP Formula, we obtain

Theorem (Weak Integration-by-Parts Characterization)

Let $K \in L^2 \times L^2$. Assume that the co-ordinate process $X\colon \Omega \to L^2$ satisfies $X \in L^2(\mathrm{d}t \otimes \mathbf{P})$, i.e.

$$\int_0^1 \mathbf{E}\left[X_t^2\right] \mathrm{d}t < \infty.$$

Then X is centered Gaussian with the Fredholm kernel K if and only if

$$\mathbf{E}\left[\int_0^1 X_t D_t f(X)\,\mathrm{d}t\right] = \mathbf{E}\left[\int_0^1\!\!\int_0^1 K(t,s) K^*\left[D^2_{t,\cdot} f(X)\right](s)\,\mathrm{d}s\mathrm{d}t\right]$$

for all $f \in \mathcal{S}$.

Thank you for listening!

Any questions?