On Skorohod-type stochastic differential equations with respect to fractional Brownian motion

Tommi Sottinen

University of Helsinki

July 17-21, 2006

Ongoing joint work with Hagen Gilsing, Humboldt University, Berlin
Outline

1. Aim
Outline

1 Aim
2 Fractional Brownian motion
Outline

1. Aim
2. Fractional Brownian motion
3. Wiener integrals
Outline

1. Aim
2. Fractional Brownian motion
3. Wiener integrals
4. Fractional Hilbert spaces
Outline

1. Aim
2. Fractional Brownian motion
3. Wiener integrals
4. Fractional Hilbert spaces
5. Malliavin calculus

Tommi Sottinen (University of Helsinki)
Skorohod-type SDE's for fBm's
July 17-21, 2006
Outline

1. Aim
2. Fractional Brownian motion
3. Wiener integrals
4. Fractional Hilbert spaces
5. Malliavin calculus
6. S-transform and Wick calculus
Outline

1. Aim
2. Fractional Brownian motion
3. Wiener integrals
4. Fractional Hilbert spaces
5. Malliavin calculus
6. $S$-transform and Wick calculus
7. Dead-end approaches
1. Aim
2. Fractional Brownian motion
3. Wiener integrals
4. Fractional Hilbert spaces
5. Malliavin calculus
6. $S$-transform and Wick calculus
7. Dead-end approaches
8. Affine equations
Outline

1. Aim
2. Fractional Brownian motion
3. Wiener integrals
4. Fractional Hilbert spaces
5. Malliavin calculus
6. S-transform and Wick calculus
7. Dead-end approaches
8. Affine equations
9. Wick equations
We consider Skorohod-type equations for fractional Brownian motion:

\[ X(t) = X_0 + \int_0^t f(r, X(r)) \, dr + \int_0^t g(r, X(r)) \, \delta B(r). \]

Here \( \delta \) denotes the Skorohod integral and \( B \) is a fractional Brownian motion.
1. Aim

- We consider Skorohod-type equations for fractional Brownian motion:

\[ X(t) = X_0 + \int_0^t f(r, X(r)) \, dr + \int_0^t g(r, X(r)) \, \delta B(r). \]

Here \( \delta \) denotes the Skorohod integral and \( B \) is a fractional Brownian motion.

- In general not much is known about such equations. Not even about the existence of the solution.
1. Aim

- We consider Skorohod-type equations for fractional Brownian motion:

\[ X(t) = X_0 + \int_0^t f(r, X(r)) \, dr + \int_0^t g(r, X(r)) \, \delta B(r). \]

Here \( \delta \) denotes the Skorohod integral and \( B \) is a fractional Brownian motion.

- In general not much is known about such equations. Not even about the existence of the solution.

- We shall use the S-transform and Wick calculus to provide explicit solutions in some special cases. Although the results are somewhat modest we believe that our approach will turn out to be useful.
2. Fractional Brownian motion

- Fractional Brownian motion $B = (B(t); t \in [0, T])$ is a centred stationary-increment Gaussian process with self-similarity property

\[ B(ct) \overset{d}{=} c^H B(t). \]

The parameter $H \in (0, 1)$ is called the Hurst index.
2. Fractional Brownian motion

- Fractional Brownian motion $B = (B(t); t \in [0, T])$ is a centred stationary-increment Gaussian process with self-similarity property

$$B(ct) \overset{d}{=} c^H B(t).$$

The parameter $H \in (0, 1)$ is called the Hurst index.

- It follows that

$$R(t, s) := \mathbb{E}[B(t)B(s)] = \frac{1}{2} \left(|t|^{2H} + |s|^{2H} - |t - s|^{2H}\right).$$
Let $\mathcal{H}_1$ be the linear space of $B$, i.e. it is the closed span of random variables $B(t)$, $t \in [0, T]$, in $L^2(\Omega)$. 

The Wiener integral is the linear continuous extension of $I(\mathbf{1}_{[0,t]}) = B(t)$.

For $f \in \mathcal{H}$, the integral is defined as $\int_0^T f(r) \, dB(r) := I(f)$. 

Tommi Sottinen (University of Helsinki)
3. Wiener integrals

- Let $\mathcal{H}_1$ be the linear space of $B$, i.e. it is the closed span of random variables $B(t)$, $t \in [0, T]$, in $L^2(\Omega)$.
- Let $\mathcal{H}$ be the closure of the linear span of indicators $1_{[0,t]}$, $t \in [0, T]$, in the inner product
  \[ \langle 1_{[0,t]}, 1_{[0,s]} \rangle = R(t, s). \]
3. Wiener integrals

- Let $\mathcal{H}_1$ be the linear space of $B$, i.e. it is the closed span of random variables $B(t)$, $t \in [0, T]$, in $L^2(\Omega)$.

- Let $\mathcal{H}$ be the closure of the linear span of indicators $\mathbf{1}_{[0,t]}$, $t \in [0, T]$, in the inner product

$$\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle = R(t, s).$$

- For $f \in \mathcal{H}$ the Wiener integral is the linear continuous extension of

$$I(\mathbf{1}_{[0,t]}) = B(t).$$

We also denote

$$\int_0^T f(r) \, dB(r) := I(f).$$
For $\alpha \in (0, 1)$ introduce the fractional integro-differential operators:

\[
I_{\alpha}^-[f](t) := \frac{1}{1-\alpha} \left( \frac{f(t)}{(T-t)^{\alpha}} + \alpha \int_t^T \frac{f(t) - f(r)}{(r-t)^{\alpha+1}} \, dr \right),
\]

\[
I_{\alpha}^+[f](t) := \frac{1}{\Gamma(\alpha)} \int_t^T f(r)(r-t)^{\alpha-1} \, dr,
\]
For $\alpha \in (0, 1)$ introduce the fractional integro-differential operators:

$$I^{\alpha}[f](t) := \frac{1}{\Gamma(\alpha)} \int_t^T f(r)(r - t)^{\alpha - 1} \, dr,$$

$$I^{-\alpha}[f](t) := \frac{1}{1 - \alpha} \left( \frac{f(t)}{(T - t)^\alpha} + \alpha \int_t^T \frac{f(t) - f(r)}{(r - t)^{\alpha + 1}} \, dr \right).$$

Define twisted integro-differential operators:

$$K[f](t) := c_H t^{\frac{1}{2} - H} I^{\frac{H - 1}{2}} ((\cdot)^{H - \frac{1}{2}} f(\cdot)) (t),$$

$$K^{-1}[f](t) := c_H^{-1} t^{\frac{1}{2} - H} I^{\frac{1}{2} - H} ((\cdot)^{H - \frac{1}{2}} f(\cdot)) (t),$$

where $c_H^2 = 2H\Gamma\left(\frac{3}{2} - H\right)/\Gamma(H + \frac{1}{2})\Gamma(2 - 2H))$. 
Now, in $\omega$-by-$\omega$ and $L^2(\Omega)$ sense we have

$$B(t) = \int_0^t K \left[ 1_{[0,t]} \right] (r) dW(r),$$

$$W(t) = \int_0^t K^{-1} \left[ 1_{[0,t]} \right] (r) dB(r),$$

where $W$ is a standard Brownian motion.
Now, in $\omega$-by-$\omega$ and $L^2(\Omega)$ sense we have

$$B(t) = \int_0^t K[1_{[0,t]}](r) dW(r),$$

$$W(t) = \int_0^t K^{-1}[1_{[0,t]}](r) dB(r),$$

where $W$ is a standard Brownian motion.

It follows that $H = K^{-1}L^2([0, T])$ and

$$\langle f, g \rangle = \int_0^T K[f](r)K[g](r) dr.$$
4. Fractional Hilbert spaces (2/2)

- Now, in $\omega$-by-$\omega$ and $L^2(\Omega)$ sense we have

$$B(t) = \int_0^t K[1_{[0,t]}](r)\,dW(r),$$

$$W(t) = \int_0^t K^{-1}[1_{[0,t]}](r)\,dB(r),$$

where $W$ is a standard Brownian motion.

- It follows that $H = K^{-1}L^2([0, T])$ and

$$\langle f, g \rangle = \int_0^T K[f](r)K[g](r)\,dr.$$

- For $H > \frac{1}{2}$ we can also write

$$\langle f, g \rangle = \int_0^T \int_0^T f(r)g(r')\phi(r, r')\,dr\,dr',$$

where $\phi = \partial^2 R/\partial r \partial r'$. 
5. Malliavin calculus

- The Malliavin derivative $D$ is defined by inverting the Wiener integral and by imposing a chain rule: If $X = F(I(f_1), \ldots, I(f_n))$ then

$$DX = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i} F(I(f_1), \ldots, I(f_n)) \cdot f_i.$$
5. Malliavin calculus

- The Malliavin derivative $D$ is defined by inverting the Wiener integral and by imposing a chain rule: If $X = F(I(f_1), \ldots, I(f_n))$ then

  $$DX = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i} F(I(f_1), \ldots, I(f_n)) \cdot f_i.$$ 

- The Skorohod integral $\delta$ is the dual operator of $D$ given by

  $$\mathbb{E} [\delta(u)X] = \mathbb{E} [\langle DX, u \rangle]$$

  for all $X$. We shall also denote

  $$\int_0^T u(r) \delta B(r) := \delta(u).$$
6. *S*-transform and Wick calculus (1/2)

- **S-transform** is a kind of infinite-dimensional Fourier transform:
  \[ S[X](\eta) := \mathbb{E} \left[ Xe^{i\eta} - \frac{1}{2}\|h\|^2 \right], \quad \eta \in \mathcal{H} \text{ and } X \in L^2(\Omega). \]
6. *S*-transform and Wick calculus (1/2)

- **S-transform** is a kind of infinite-dimensional Fourier transform:
  \[ S[X](\eta) := \mathbb{E} \left[ Xe^{\frac{1}{2}\|h\|^2} \right], \quad \eta \in \mathcal{H} \text{ and } X \in L^2(\Omega). \]

- **S-transform** is injective and \( S[B(t)](\eta) = \int_0^t \eta(r) \, dr. \)
6. S-transform and Wick calculus (1/2)

- **S-transform** is a kind of infinite-dimensional Fourier transform:

\[ S[X](\eta) := \mathbb{E} \left[ X e^{i \eta} \right] - \frac{1}{2} \| h \|^2 , \quad \eta \in \mathcal{H} \text{ and } X \in L^2(\Omega). \]

- S-transform is injective and \( S[B(t)](\eta) = \int_0^t \eta(r) \, dr \).

- **Wick product** is defined by \( S[X \diamond Y](\eta) = S[X](\eta) \cdot S[Y](\eta) \).
S-transform is a kind of infinite-dimensional Fourier transform:

\[ S[X](\eta) := \mathbb{E}\left[X e^{i\eta - \frac{1}{2}\|\eta\|^2}\right], \quad \eta \in \mathcal{H} \text{ and } X \in L^2(\Omega). \]

- S-transform is injective and \( S[B(t)](\eta) = \int_0^t \eta(r) \, dr \).
- Wick product is defined by \( S[X \diamond Y](\eta) = S[X](\eta) \cdot S[Y](\eta) \).
- S-transform depends on \( B \) but the Wick product does not.
6. **S-transform and Wick calculus**  

- **S-transform** is a kind of infinite-dimensional Fourier transform:
  \[
  S[X](\eta) := E \left[ X e^{i(\eta) - \frac{1}{2} \|h\|^2} \right], \quad \eta \in \mathcal{H} \text{ and } X \in L^2(\Omega).
  \]

- **S-transform** is injective and \( S[B(t)](\eta) = \int_0^t \eta(r) \, dr \).

- **Wick product** is defined by \( S[X \diamond Y](\eta) = S[X](\eta) \cdot S[Y](\eta) \).

- **S-transform** depends on \( B \) but the **Wick product** does not.

- **Wick power** is \( X^{\diamond 0} := 1 \) and \( X^{\diamond(n+1)} := X^{\diamond n} \diamond X \).
**S-transform** is a kind of infinite-dimensional Fourier transform:

\[ S[X](\eta) := \mathbb{E} \left[ X e^{i(\eta) - \frac{1}{2}\|h\|^2} \right], \quad \eta \in \mathcal{H} \text{ and } X \in L^2(\Omega). \]

- **S-transform** is injective and \( S[B(t)](\eta) = \int_0^t \eta(r) \, dr \).
- **Wick product** is defined by \( S[X \diamond Y](\eta) = S[X](\eta) \cdot S[Y](\eta) \).
- **S-transform** depends on \( B \) but the Wick product does not.
- **Wick power** is \( X^{\diamond 0} := 1 \) and \( X^{\diamond (n+1)} := X^{\diamond n} \diamond X \).
- **Wick exponent** is \( e^{\diamond X} := \sum_{n=0}^{\infty} \frac{1}{n!} X^{\diamond n} \).
6. \textit{S}-transform and Wick calculus (1/2)

- \textbf{S-transform} is a kind of infinite-dimensional Fourier transform:

\[
S[X](\eta) := \mathbb{E} \left[ X e^{i(\eta) - \frac{1}{2} \|h\|^2} \right], \quad \eta \in \mathcal{H} \text{ and } X \in L^2(\Omega).
\]

- S-transform is injective and \( S[B(t)](\eta) = \int_0^t \eta(r) \, dr. \)

- Wick product is defined by \( S[X \diamond Y](\eta) = S[X](\eta) \cdot S[Y](\eta). \)

- S-transform depends on \( B \) but the Wick product does not.

- Wick power is \( X^{\diamond 0} := 1 \) and \( X^{\diamond (n+1)} := X^{\diamond n} \diamond X. \)

- Wick exponent is \( e^{\diamond X} := \sum_{n=0}^{\infty} \frac{1}{n!} X^{\diamond n}. \)

- Hermite polynomial is \( H_n(x) := \frac{(-1)^n}{n!} e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}}. \)
6. S-transform and Wick calculus (2/2)

- Some rules:
Some rules:

1. \[ e^{\diamond I(f)} = e^I(f) - \frac{1}{2} \| f \|^2, \]
Some rules:

1. $e^{\diamond I(f)} = e^{I(f)} - \frac{1}{2} \|f\|^2$,
2. $e^{\diamond I(f)} \diamond e^{\diamond I(g)} = e^{\diamond I(f+g)}$, 

$\int_{0}^{t} u(r) \delta B(r) = \lim_{n \to \infty} \sum_{r_i \in \pi_n} u(r_i - 1) \diamond (B(r_i) - B(r_i - 1))$.

$S[\int_{0}^{t} u(r) \delta B(r)](\eta) = \int_{0}^{t} S[u(r)](\eta) \eta(r) \, dr$. 

Tommi Sottinen (University of Helsinki)  
Skorohod-type SDE’s for fBm’s  
July 17-21, 2006
Some rules:

1. $e^{\diamond l(f)} = e^{l(f)} - \frac{1}{2} \|f\|^2$,
2. $e^{\diamond l(f)} \diamond e^{\diamond l(g)} = e^{\diamond l(f+g)}$,
3. $l(f)^n = \|f\|^n H_n(l(f))$. 
6. $S$-transform and Wick calculus (2/2)

- Some rules:
  1. $e^{\diamond I(f)} = e^{I(f)} - \frac{1}{2} \|f\|^2$,
  2. $e^{\diamond I(f)} \diamond e^{\diamond I(g)} = e^{\diamond I(f+g)}$,
  3. $I(f)^{\diamond n} = \|f\|^n H_n(I(f))$.

- Connection between Wick product and Skorohod integral:

\[
\int_0^t u(r) \delta B(r) = \lim_{n \to \infty} \sum_{r_i \in \pi_n} u(r_{i-1}) \diamond (B(r_i) - B(r_{i-1})).
\]
Some rules:

1. \( e^{\diamond I(f)} = e^{I(f)} - \frac{1}{2} \|f\|^2 \),
2. \( e^{\diamond I(f)} \diamond e^{\diamond I(g)} = e^{\diamond I(f+g)} \),
3. \( I(f)^{\diamond n} = \|f\|^n H_n(I(f)) \).

Connection between Wick product and Skorohod integral:

\[
\int_0^t u(r) \delta B(r) = \lim_{n \to \infty} \sum_{r_i \in \pi_n} u(r_{i-1}) \diamond (B(r_i) - B(r_{i-1})).
\]

Connection between \( S\)-transform and Skorohod integral:

\[
S \left[ \int_0^t u(r) \delta B(r) \right](\eta) = \int_0^t S[u(r)](\eta)\eta(r) \, dr.
\]
7. Dead-end approaches (1/2)

- **Picard iteration**: We have only bounds with Malliavin derivative for the Skorohod integral. The problem is that we get a vicious loop:

\[
D_s[X(t)] = D_s[X_0] + \int_0^t \frac{\partial f}{\partial x}(r, X(r)) D_s[X(r)] \, dr \\
+ g(s, X(s)) + \int_0^t \frac{\partial g}{\partial x}(r, X(r)) D_s[X(r)] \delta B(r).
\]
7. Dead-end approaches (1/2)

- **Picard iteration:** We have only bounds with Malliavin derivative for the Skorohod integral. The problem is that we get a vicious loop:

\[
D_s[X(t)] = D_s[X_0] + \int_0^t \frac{\partial f}{\partial x}(r, X(r)) D_s[X(r)] \, dr \\
+ g(s, X(s)) + \int_0^t \frac{\partial g}{\partial x}(r, X(r)) D_s[X(r)] \, \delta B(r).
\]

- **Forward integrals:** For \( H > \frac{1}{2} \) we can write

\[
X(t) = X_0 + \int_0^t f(r, X(r)) \, dr + \int_0^t g(r, X(r)) \, dB(r) \\
+ \int_0^t \int_0^t \frac{\partial g}{\partial x}(r, X(r)) D_{r'}[X(r)] \phi(r, r') \, dr \, dr'.
\]

Here the forward integral \( dB(r) \) is nice, but the correction term leads to a vicious loop, as before.
Transfer principle: We can write the equation with respect to a standard Brownian motion

\[ X(t) = X_0 + \int_0^t f(r, X(r)) \, dr + \int_0^t K[g(\cdot, X(\cdot))] (r) \delta W(r). \]

But \( K \) “looks into the future”. So, this approach does not seem to be very useful.
Consider the *affine equation*

\[ X(t) = X_0 + \int_0^t f_0(r) + f_1(r)X(r) \, dr + \int_0^t g_0(r) + g_1(r)X(r) \, \delta B(r), \]

with \( f_0, f_1, g_0, g_1 \in \mathcal{H}. \)
Consider the affine equation

\[ X(t) = X_0 + \int_0^t f_0(r) + f_1(r)X(r) \, dr + \int_0^t g_0(r) + g_1(r)X(r) \, \delta B(r), \]

with \( f_0, f_1, g_0, g_1 \in \mathcal{H} \).

The solution to the affine equation is

\[
X(t) = e^{\langle f_1 \rangle t} X_0 + \int_0^t e^{\langle f_1 \rangle r} f_1(r) \, dr + \int_0^t e^{\langle g_1 \rangle r} g_1(r) \, \delta B(r) \\
+ \left[ X_0 + \int_0^t e^{\langle f_1 \rangle r} f_1(r) \, dr - \int_0^r e^{\langle g_1 \rangle r'} g_1(r') \, \delta B(r') \right] f_0(r) \, dr \\
+ \int_0^t e^{\langle f_1 \rangle r} f_1(r) \, dr - \int_0^r e^{\langle g_1 \rangle r'} g_1(r') \, \delta B(r') \right] g_0(r) \, \delta B(r)
\]

where \( \|g\|^2(r, t) := \|g1_{[r,t]}\|^2 = \int_r^t K[g](r')^2 \, dr \).
If $X_0$ is deterministic we can eliminate the Wick products:

\[
X(t) = e^{\int_0^t f_1(r) dr + \int_0^t g_1(r) \delta B(r)} - \frac{1}{2} \|g_1\|^2(0,t) X_0
+ \int_0^t e^{\int_r^t f_1(r') dr'} + \int_r^t g_1(r') \delta B(r') - \frac{1}{2} \|g_1\|^2(r,t) f_0(r) \, dr
+ \int_0^t e^{\int_r^t f_1(r') dr'} + \int_r^t g_1(r') \delta B(r') - \frac{1}{2} \left(\|g_1\|^2(0,t) + \|g_1\|^2(0,r)\right) g_0(r) \delta B(r).
\]
If $X_0$ is deterministic we can eliminate the Wick products:

\[
X(t) = e^{\int_0^t f_1(r) dr + \int_0^t g_1(r) \delta B(r)} - \frac{1}{2} \|g_1\|^2(0,t) X_0
\]

\[
+ \int_0^t e^{\int_r^t f_1(r') dr'} + \int_r^t g_1(r') \delta B(r') - \frac{1}{2} \|g_1\|^2(r,t) f_0(r) dr
\]

\[
+ \int_0^t e^{\int_r^t f_1(r') dr'} + \int_r^t g_1(r') \delta B(r') - \frac{1}{2} \left( \|g_1\|^2(0,t) + \|g_1\|^2(0,r) \right) g_0(r) \delta B(r).
\]

Moreover, if $H > \frac{1}{2}$ the Skorohod integral can be transferred to a forward integral.
If $X_0$ is deterministic we can eliminate the Wick products:

\[
X(t) = e^{\int_0^t f_1(r)dr + \int_0^t g_1(r)\delta B(r)} - \frac{1}{2} \|g_1\|^2(0,t) X_0 \\
+ \int_0^t e^{\int_r^t f_1(r')dr'} + \int_r^t g_1(r')\delta B(r') - \frac{1}{2} \|g_1\|^2(t,r) f_0(r) dr \\
+ \int_0^t e^{\int_r^t f_1(r')dr'} + \int_r^t g_1(r')\delta B(r') - \frac{1}{2} \left( \|g_1\|^2(0,t) + \|g_1\|^2(0,r) \right) g_0(r) \delta B(r).
\]

Moreover, if $H > \frac{1}{2}$ the Skorohod integral can be transfered to a forward integral.

**Idea of proof:** Use the $S$-transform to get an ordinary non-homogeneous linear equation for $y(t) = S[X(t)](\eta)$ and solve it.
Suppose \( f = f(t, x) \) is entire in \( x \), i.e. \( f(t, x) = \sum_{n=0}^{\infty} f_n(t) x^n \). The Wick function associated to \( f \) and a random variable \( X \) is

\[
f^\diamond(t, X) := \sum_{n=0}^{\infty} f_n(t) X^{\diamond n}.
\]
Suppose $f = f(t, x)$ is entire in $x$, i.e. $f(t, x) = \sum_{n=0}^{\infty} f_n(t)x^n$. The Wick function associated to $f$ and a random variable $X$ is

$$f^\diamond(t, X) := \sum_{n=0}^{\infty} f_n(t)X^n.$$ 

Consider the Wick equation

$$X(t) = X_0 + \int_0^t f^\diamond(r, X(r)) \, dr + \int_0^t g^\diamond(r, X(r)) \, \delta B(r).$$

Suppose that there exists $h = h(t, z)$ such that

$$\frac{\partial h}{\partial t}(t, z) = f(t, h(t, z)) \quad \text{and} \quad \frac{\partial h}{\partial z}(t, z) = g(t, h(t, z)).$$

Assume further that $h$ is entire in $z$. 
The solution to the Wick equation is

\[ X(t) = h^\diamond(t, B(t)) = \sum_{n=0}^{\infty} h_n(t) t^{2H} H_n(B(t)). \]

The \( h_n(t) \) can be solved iteratively from the coefficients \( f_n(t) \) and \( g_n(t) \).
The solution to the Wick equation is

\[ X(t) = h^\Diamond(t, B(t)) = \sum_{n=0}^{\infty} h_n(t) t^{2H} H_n(B(t)). \]

The \( h_n(t) \) can be solved iteratively from the coefficients \( f_n(t) \) and \( g_n(t) \).

Idea of proof: With the \( S \)-transform we have

\[ S[f^\Diamond(r, (X(r)))](\eta) = f(r, S[X(r)](\eta)). \]

So, we just need to find an entire solution to the ODE

\[ y(t) = f(t, y(t)) + g(t, y(t))\eta(t) \]

of the type \( y(t) = h(t, \int_{0}^{t} \eta(r) \, dr) \). Then \( h^\Diamond(t, B(t)) \) solves the Wick equation.