On Skorohod-type stochastic differential equations with respect to fractional Brownian motion

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July 17-21,2006

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Outline

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9. Wick equations

1. Aim

- We consider Skorohod-type equations for fractional Brownian motion:

$$X(t) = X_0 + \int_0^t f(r, X(r))\,\mathrm{d}r + \int_0^t g(r, X(r))\,\delta B(r).$$

Here δ denotes the Skorohod integral and B is a fractional Brownian motion.

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  Here δ denotes the Skorohod integral and B is a fractional Brownian motion.
- In general not much is known about such equations. Not even about the existence of the solution.
- We shall use the S-transform and Wick calculus to provide explicit solutions in some special cases. Although the results are somewhat modest we believe that our approach will turn out to be useful.

2. Fractional Brownian motion

- Fractional Brownian motion $B = (B(t); t \in [0, T])$ is a centred stationary-increment Gaussian process with self-similarity property

$$B(ct) \stackrel{d}{=} c^H B(t).$$

The parameter $H \in (0, 1)$ is called the Hurst index.

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- It follows that

$$R(t,s) := \mathbf{E}[B(t)B(s)] = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right).$$

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3. Wiener integrals

- Let $\mathcal{H}_1$ be the linear space of B, i.e. it is the closed span of random variables $B(t)$, $t \in [0, T]$, in $L^2(\Omega)$.

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- Let $\mathcal{H}$ be the closure of the linear span of indicators $\mathbf{1}_{[0,t]}$, $t \in [0, T]$, in the inner product

$$\left\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \right\rangle = R(t, s).$$

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$$\left\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \right\rangle = R(t, s).$$

- For $f \in \mathcal{H}$ the Wiener integral is the linear continuous extension of

$$I(\mathbf{1}_{[0,t]}) = B(t).$$

We also denote

$$\int_0^T f(r)\,\mathrm{d}B(r) := I(f).$$

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- For $\alpha \in (0,1)$ introduce the fractional integro-differential operators:

$$
\begin{aligned}
I_{-}^{\alpha}[f](t) &:= \frac{1}{\Gamma(\alpha)} \int_t^T f(r)(r-t)^{\alpha-1}\,\mathrm{d}r, \\
I_{-}^{-\alpha}[f](t) &:= \frac{1}{1-\alpha}\left(\frac{f(t)}{(T-t)^{\alpha}} + \alpha \int_t^T \frac{f(t)-f(r)}{(r-t)^{\alpha+1}}\,\mathrm{d}r \right).
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- Define twisted integro-differential operators:

$$\mathrm{K}[f](t) := c_H t^{\frac{1}{2}-H} I_-^{H-\frac{1}{2}}\left[(\cdot)^{H-\frac{1}{2}} f(\cdot)\right](t),$$
$$\mathrm{K}^{-1}[f](t) := c_H^{-1} t^{\frac{1}{2}-H} I_-^{\frac{1}{2}-H}\left[(\cdot)^{H-\frac{1}{2}} f(\cdot)\right](t),$$

where $c_H^2 = 2H\Gamma(\frac{3}{2}-H)/(\Gamma(H+\frac{1}{2})\Gamma(2-2H))$.

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- Now, in ω-by-ω and $L^2(\Omega)$ sense we have

$$\begin{aligned} B(t) &= \int_0^t \mathrm{K}\left[\mathbf{1}_{[0,t]}\right](r)\,\mathrm{d}W(r), \\ W(t) &= \int_0^t \mathrm{K}^{-1}\left[\mathbf{1}_{[0,t]}\right](r)\,\mathrm{d}B(r), \end{aligned}$$

where W is a standard Brownian motion.

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- It follows that $\mathcal{H} = \mathrm{K}^{-1}L^2([0,T])$ and

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- For $H > \frac{1}{2}$ we can also write

$$\langle f, g\rangle = \int_0^T \int_0^T f(r)g(r')\phi(r,r')\,\mathrm{d}r\mathrm{d}r',$$

where $\phi = \partial^2 R/\partial r \partial r'$.

5. Malliavin calculus

- The Malliavin derivative D is defined by inverting the Wiener integral and by imposing a chain rule: If $X = F(I(f_1), \ldots, I(f_n))$ then

$$DX = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i} F(I(f_1), \ldots, I(f_n)) \cdot f_i.$$

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- The Skorohod integral δ is the dual operator of D given by

$$\mathbf{E}\left[\delta(u)X\right] = \mathbf{E}\left[\langle DX, u\rangle\right]$$

for all X. We shall also denote

$$\int_0^T u(r)\,\delta B(r) := \delta(u).$$

6. S-transform and Wick calculus (1/2)

- S-transform is a kind of infinite-dimensional Fourier transform:

$$S[X](\eta) := \mathbf{E}\left[Xe^{I(\eta)-\frac{1}{2}\|h\|^2}\right], \quad \eta \in \mathcal{H} \text{ and } X \in L^2(\Omega).$$

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- Wick product is defined by $S[X \diamond Y](\eta) = S[X](\eta) \cdot S[Y](\eta)$.

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- Wick exponent is $e^{\diamond X} := \sum_{n=0}^{\infty} \frac{1}{n!} X^{\diamond n}$.
- Hermite polynomial is $H_n(x) := \frac{(-1)^n}{n!} e^{\frac{x^2}{2}} \frac{\mathrm{d}^n}{\mathrm{d}x^n} e^{-\frac{x^2}{2}}$.

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6. S-transform and Wick calculus (2/2)

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- Connection between Wick product and Skorohod integral:

$$\int_0^t u(r)\,\delta B(r) = \lim_{n\to\infty} \sum_{r_i \in \pi_n} u(r_{i-1}) \diamond (B(r_i) - B(r_{i-1}))\,.$$

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- Connection between S-transform and Skorohod integral:

$$S\left[\int_0^t u(r)\,\delta B(r)\right](\eta) = \int_0^t S[u(r)](\eta)\eta(r)\,\mathrm{d}r.$$

7. Dead-end approaches (1/2)

- Picard iteration: We have only bounds with Malliavin derivative for the Skorohod integral. The problem is that we get a vicious loop:

$$
\begin{aligned}
D_s[X(t)] &= D_s[X_0] + \int_0^t \frac{\partial f}{\partial x}(r, X(r)) D_s[X(r)] \, \mathrm{d}r \\
&\quad + g(s, X(s)) + \int_0^t \frac{\partial g}{\partial x}(r, X(r)) D_s[X(r)] \, \delta B(r).
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- Forward integrals: For $H > \frac{1}{2}$ we can write

$$\begin{aligned} X(t) &= X_0 + \int_0^t f(r, X(r))\,\mathrm{d}r + \int_0^t g(r, X(r))\,\mathrm{d}B(r) \\ &\quad + \int_0^t \int_0^t \frac{\partial g}{\partial x}(r, X(r)) D_{r'}[X(r)]\phi(r, r')\,\mathrm{d}r\mathrm{d}r'. \end{aligned}$$

Here the forward integral $\mathrm{d}B(r)$ in nice, but the correction term leads to a vicious loop, as before.

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- Transfer principle: We can write the equation with respect to a standard Brownian motion

$$X(t) = X_0 + \int_0^t f(r, X(r))\,\mathrm{d}r + \int_0^t \mathrm{K}[g(\cdot, X(\cdot))](r)\,\delta W(r).$$

But K "looks into the future". So, this approach does not seem to be very useful.

8. Affine equations (1/2)

Consider the affine equation

$$X(t) = X_0 + \int_0^t f_0(r) + f_1(r)X(r)\,\mathrm{d}r + \int_0^t g_0(r) + g_1(r)X(r)\,\delta B(r),$$

with $f_0, f_1, g_0, g_1 \in \mathcal{H}$.

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with $f_0, f_1, g_0, g_1 \in \mathcal{H}$.

The solution to the affine equation is

$$\begin{aligned} X(t) &= e^{\diamond \int_0^t f_1(r)\mathrm{d}r + \int_0^t g_1(r)\delta B(r)} \\ &\quad \diamond \left(X_0 + \int_0^t e^{\diamond - \int_0^r f_1(r')\mathrm{d}r' - \int_0^r g_1(r')\delta B(r')} f_0(r)\,\mathrm{d}r \right. \\ &\qquad \left. + \int_0^t e^{\diamond - \int_0^r f_1(r')\mathrm{d}r' - \int_0^r g_1(r')\delta B(r')} g_0(r)\,\delta B(r) \right), \end{aligned}$$

where $\|g\|^2(r,t) := \|g\mathbf{1}_{[r,t]}\|^2 = \int_r^t \mathrm{K}[g](r')^2\,\mathrm{d}r.$

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8. Affine equations (2/2)

If X_0 is deterministic we can eliminate the Wick products:

$$
\begin{aligned}
X(t) &= e^{\int_0^t f_1(r)\mathrm{d}r+\int_0^t g_1(r)\delta B(r)-\frac{1}{2}\|g_1\|^2(0,t)} X_0 \\
&\quad + \int_0^t e^{\int_r^t f_1(r')\mathrm{d}r'+\int_r^t g_1(r')\delta B(r')-\frac{1}{2}\|g_1\|^2(r,t)} f_0(r)\,\mathrm{d}r \\
&\quad + \int_0^t e^{\int_r^t f_1(r')\mathrm{d}r'+\int_r^t g_1(r')\delta B(r')-\frac{1}{2}\left(\|g_1\|^2(0,t)+\|g_1\|^2(0,r)\right)} g_0(r)\,\delta B(r).
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&+\int_0^t e^{\int_r^t f_1(r')\mathrm{d}r'+\int_r^t g_1(r')\delta B(r')-\frac{1}{2}\left(\|g_1\|^2(0,t)+\|g_1\|^2(0,r)\right)}g_0(r)\,\delta B(r).
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Moreover, if $H > \frac{1}{2}$ the Skorohod integral can be transfered to a forward integral.

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\end{aligned}
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Moreover, if $H > \frac{1}{2}$ the Skorohod integral can be transfered to a forward integral.

Idea of proof: Use the S-transform to get an ordinary non-homogeneous linear equation for $y(t) = S[X(t)](\eta)$ and solve it.

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9. Wick equations (1/2)

Suppose $f = f(t,x)$ is entire in x, i.e. $f(t,x) = \sum_{n=0}^{\infty} f_n(t)x^n$. The Wick function associated to f and a random variable X is

$$f^{\diamond}(t,X) := \sum_{n=0}^{\infty} f_n(t) X^{\diamond n}.$$

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Consider the Wick equation

$$X(t) = X_0 + \int_0^t f^{\diamond}(r, X(r))\,\mathrm{d}r + \int_0^t g^{\diamond}(r, X(r))\,\delta B(r).$$

Suppose that there exists $h = h(t,z)$ such that

$$\frac{\partial h}{\partial t}(t,z) = f(t, h(t,z)) \quad \text{and} \quad \frac{\partial h}{\partial z}(t,z) = g(t, h(t,z)).$$

Assume further that h is entire in z.

9. Wick equations (2/2)

The solution to the Wick equation is

$$X(t) = h^{\diamond}(t, B(t)) = \sum_{n=0}^{\infty} h_n(t) t^{2H} H_n(B(t)).$$

The $h_n(t)$ can be solved iteratively from the coefficients $f_n(t)$ and $g_n(t)$.

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The $h_n(t)$ can be solved iteratively from the coefficients $f_n(t)$ and $g_n(t)$.

Idea of proof: With the S-transform we have

$$S[f^{\diamond}(r, (X(r))](\eta) = f(r, S[X(r)](\eta)).$$

So, we just need to find an entire solution to the ODE

$$y(t) = f(t, y(t)) + g(t, y(t))\eta(t)$$

of the type $y(t) = h(t, \int_0^t \eta(r)\mathrm{d}r)$. Then $h^{\diamond}(t, B(t))$ solves the Wick equation.