Integration-by-Parts Characterizations of Gaussian Processes

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References

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Abstract

The Stein's lemma characterizes the Gaussian distribution via an integration-by-parts formula.

We show that a similar integration-by-parts formula characterizes a wide class of Gaussian processes, the so-called Gaussian Fredholm processes. These processes include rough long-range dependent fractional processes like the fractional Brownian motions.

Outline

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Stein's (Multivariate) Lemma

Stein's Lemma, a.k.a. the Integration-by-Parts Characterization, states that a random variable X is standard normal if and only if

$$\mathsf{E}\left[Xf'(X)\right] = \mathsf{E}\left[f''(X)\right]$$

for all smooth and bounded enough $f : \mathbb{R} \to \mathbb{R}$.

Multivariate Stein's Lemma states that $X = (X_1, \ldots, X_d)$ is centered Gaussian with covariance R if and only if

$$\mathsf{E}\left[\sum_{i=1}^{d} X_i \frac{\partial}{\partial x_i} f(X)\right] = \mathsf{E}\left[\sum_{i=1}^{d}\sum_{j=1}^{d} R_{ij} \frac{\partial^2}{\partial x_i \partial x_j} f(X)\right]$$

for all smooth and bounded enough $f : \mathbb{R}^d \to \mathbb{R}$

Stein's (Multivariate) Lemma

Let $X = (X_t)_{t \in [0,1]}$ be a centered process with covariance R. The Multivariate Stein's Lemma suggests us to guess (WRONGLY!) that X is Gaussian if and only if

$$\mathsf{E}\left[\int_0^1 X_t D_t f(X)\,\mathrm{d}t\right] = \mathsf{E}\left[\int_0^1 \int_0^1 R(t,s) D^2_{t,s} f(X)\,\mathrm{d}s\mathrm{d}t\right],$$

where D is some kind of Malliavin Derivative.

Outline

Theorem (Fredholm Representation)

Let $X = (X_t)_{t\in[0,1]}$ be a separable centered Gaussian process. Assume the very mild trace condition

$$\int_0^1 R(t,t)\,\mathrm{d}t < \infty.$$

Then there exists a kernel $K \in L^2 \times L^2 = L^2([0,1]^2)$ and a Brownian motion $W = (W_t)_{t\in[0,1]}$ such that

$$X_t \stackrel{d}{=} \int_0^1 K(t,s)\,\mathrm{d}W_s,$$

where d stands for equality in law in L^2.

Outline

Pathwise Malliavin Differentiation

Let $\mathcal{C}_p^\infty(\mathbb{R}^n)$ denote the space of all polynomially bounded functions with polynomially bounded partial derivatives of all orders. Consider functionals $f\colon L^2 \to \mathbb{C}$ of the form

$$f(x) = g\left(z_1(x), \ldots, z_n(x)\right),$$

where $n \in \mathbb{N}$ and $g \in \mathcal{C}_p^\infty(\mathbb{R}^n)$, and

$$z_k(x) = \int_0^1 e_k(t)\,\mathrm{d}x(t)$$

for some step functions $e_k \in \mathcal{E}$. For such f we write $f \in \mathcal{S}$.

The pathwise Malliavin derivative of such $f \in \mathcal{S}$ is

$$D_t f(x) = \sum_{k=1}^n \frac{\partial}{\partial z_k} g(z_1(x), \ldots, z_n(x))\, e_k(t).$$

Pathwise Malliavin Differentiation

More generally, by iteration for every $m \in \mathbb{N}$, the pathwise Malliavin derivative of order m is defined as follows: for every $t_1, \dots, t_m \in [0,1]$,

$$\begin{aligned} & D^m_{t_m,\dots,t_1} f(x) \\ & \quad = \sum_{1 \le k_1,\dots,k_m \le n} \frac{\partial^m}{\partial z_{k_1} \cdots \partial z_{k_m}} g(z_{k_1}(x), \dots, z_{k_n}(x)) \\ & \qquad \times \left(e_{k_1} \otimes \cdots \otimes e_{k_m}\right)(t_1, \dots, t_m). \end{aligned}$$

Remark

Let $f \in \mathcal{S}$ and $y \in L^2$. Let ∇ be the Fréchet derivative. Let $\mathrm{I}y(t) = \int_0^t y(s)\mathrm{d}s$. Then

$$\langle \nabla f(x), \mathrm{I}y \rangle_{L^2} = \langle Df(x), y \rangle_{L^2}$$

Outline

Strong Form Integration-by-Parts Characterization

Let $1_t = 1_{[0,t)}$. Let K^* extend linearly $K^* 1_t(\cdot) = K(t, \cdot)$.

Remark

$$\int_0^1 K^* f(t) g(t) \, \mathrm{d}t = \int_0^1 f(t) K g(\mathrm{d}t),$$

where

$$K g(t) = \int_0^1 g(s) K(t, s) \, \mathrm{d}s.$$

Example

If $K(\cdot, s)$ has bounded variation and f is nice enough, then

$$K^* f(s) = \int_0^1 f(t) K(\mathrm{d}t, s).$$

Strong Form Integration-by-Parts Characterization

Theorem

Let $K \in L^2 \times L^2$. The co-ordinate process $X\colon \Omega \to L^2$ is centered Gaussian with Fredholm kernel K if and only if

$$\mathsf{E}\left[X_t D_t f(X)\right] = \mathsf{E}\left[\int_0^1 K(t,s) K^*\left[D^2_{t,\cdot} f(X)\right](s)\,\mathrm{d}s\right]$$

for all $t \in [0,1]$ and $f \in \mathcal{S}$.

Outline

Weak Form Integration-by-Parts Characterization

By using Fubini to the Strong IBP Formula, we obtain

Theorem (Weak Integration-by-Parts Characterization)

Let $K \in L^2 \times L^2$. Assume that the co-ordinate process $X : \Omega \to L^2$ satisfies $X \in L^2(\mathrm{d}t \otimes \mathsf{P})$, i.e.

$$\int_0^1 \mathsf{E}\left[X_t^2\right] \mathrm{d}t < \infty.$$

Then X is centered Gaussian with the Fredholm kernel K if and only if

$$\mathsf{E}\left[\int_0^1 X_t D_t f(X)\,\mathrm{d}t\right] = \mathsf{E}\left[\int_0^1\!\!\int_0^1 K(t,s) K^*\left[D^2_{t,\cdot} f(X)\right](s)\,\mathrm{d}s\mathrm{d}t\right]$$

for all $f \in \mathcal{S}$.

Thank you for listening!

Any questions?