

WHAT IS THE PRICE OF THE FUTURE?

AN UNFINISHED OPERA IN FIVE ACTS

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DISCLAIMER

In this talk I try to follow the **EINSTEIN'S MAXIM**:

Things should be made as simple as possible, but not simpler.

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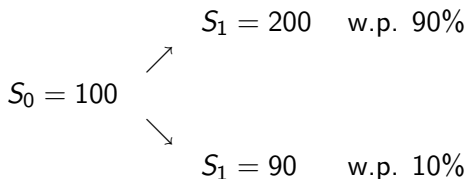
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This means that, as far as technical details are concerned, I will cheat!

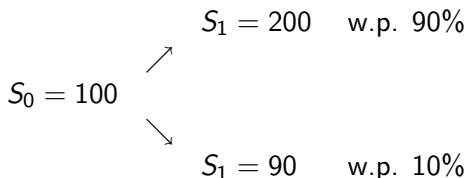
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- Mr. K. offers a European Call-Option: We get a right to buy tomorrow the stock S with today's price 100 ISK: Formula for our profit is

$$f(S_1) = (S_1 - 100)^+,$$

where $x^+ := \max(x, 0)$.

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- But if we pay 90 ISK, Mr. K. can do a follows: He takes a bank loan of 10 ISK and buys one stock (he already got 90 ISK by selling the call-option). Now, if the stock price will go up, Mr. K. gives us the stock in return of 100 ISK. After paying his bank loan Mr. K. has made a profit of 90 ISK. If, on the other hand, the stock will go down we will not exercise our option. Now Mr. K. sells his stock in the market and after paying his bank loan he has made a profit of 80 ISK.

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- By the way: The correct price in this case is 100/11 ISK. If the price is higher then Mr. K. can generate arbitrage; if the price is lower then we, the buyer, can generate arbitrage.

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- By the way: The correct price in this case is 100/11 ISK. If the price is higher then Mr. K. can generate arbitrage; if the price is lower then we, the buyer, can generate arbitrage.
- Actually, the arbitrage-free price is independent of the probabilities of stock going up or down. This should be clear from the fact that Mr. K.'s arbitrage strategy was independent of probabilities.

- 1 TRADING STRATEGIES
- 2 PRICING PRINCIPLES: REPLICATION AND NO-ARBITRAGE
- 3 FUNDAMENTAL THEOREMS OF ASSET PRICING
- 4 REPLICATION IN THE BLACK-SCHOLES MODEL
- 5 WHY QUADRATIC VARIATION, OR BROWNIAN MOTION

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- The **STOCK** $S = (S_t)_{t \in [0, T]}$ is the risky asset. It is random: The prices S_t are not known before time t .

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DEFINITION (TRADING STRATEGY)

TRADING STRATEGY is $(\pi_t)_{t \in [0, T]} = (\beta_t, \gamma_t)_{t \in [0, T]}$. Here β_t tells the amount of bonds and γ_t the amount of stocks the investor has in her portfolio at time t . The amounts β_t and γ_t can depend on the price process S up to time t , but not on the future prices of S .

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DEFINITION (VALUE OF A SELF-FINANCING STRATEGY)

The **VALUE** of a trading strategy π at time t is $V_t(\pi) = \beta_t + \gamma_t S_t$. Trading strategy is **SELF-FINANCING** if

$$dV_t(\pi) = \gamma_t dS_t. \quad (1)$$

REMARK (SELF-FINANCING AND BUDGET CONSTRAINTS)

- In (1) the differentials are understood limits in the forward sense: $dS_t \approx S_{t+\Delta} - S_t$ when $\Delta > 0$ is small.

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- Equation (1) is actually a budget constraint. Indeed, consider it in discrete time points $t < t + 1$. Then says

$$V_{t+1}(\pi) - V_t(\pi) = \gamma_t (S_{t+1} - S_t),$$

which is actually equivalent to

$$\beta_t + \gamma_t S_t = \beta_{t+1} + \gamma_{t+1} S_t.$$

This means that the value of the portfolio remain unchanged when the portfolio is rebalanced and all the changes in the value come from the changes of S .

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PRICING PRINCIPLES: REPLICATION AND NO-ARBITRAGE

DEFINITION (OPTION)

An **OPTION** is simply a function of the underlying stock. We consider here only **EUROPEAN VANILLA OPTIONS**, i.e. options that are of the form $f(S_T)$, where $f : \mathbb{R} \rightarrow \mathbb{R}$ is some function.

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DEFINITION (REPLICATION PRINCIPLE)

The **REPLICATION PRICE** of $f(S_T)$ is the initial capital $V_0(\pi)$ needed to construct a self-financing trading strategy π for which

$$V_T(\pi) = f(S_T).$$

Here π is the **REPLICATING PORTFOLIO**.

PRICING PRINCIPLES: REPLICATION AND NO-ARBITRAGE

REMARK (REPLICATION)

Suppose we can write

$$f(S_T) = c + \int_0^T \xi_t dS_t. \quad (2)$$

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Then c is the price of the option and the replicating portfolio π is determined by: $\gamma_t = \xi_t$, $\beta_0 + \gamma_0 S_0 = c$, and β_t is determined from these and the fact that the portfolio is self-financing.

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The key question is: Can one construct replicating portfolios, i.e. representation of type (2) (in theory, in practise)?

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An **ARBITRAGE** is a trading strategy π with $V_0(\pi) = 0$, $V_t(\pi) \geq 0$ for all t and $\mathbf{P}[V_T(\pi) > 0] > 0$.

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Let $P = (P_t)_{t \in [0, T]}$ be a price process for the option $f(S_T)$, i.e. P_t is the price of the option $f(S_T)$ at time t . (Obviously $P_T = f(S_T)$, but it is P_0 that we are interested in.)

PRICING PRINCIPLES: REPLICATION AND NO-ARBITRAGE

DEFINITION (NO-ARBITRAGE PRINCIPLE)

P_0 is an arbitrage-free price for $f(S_T)$ if there is no such trading strategy $\pi = (\beta, \gamma, \delta)$ satisfying the self-financing condition

$$dV_t(\pi) = \gamma_t dS_t + \delta_t dP_t$$

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REMARK (ARBITRAGE SPREAD)

The problem with the no-arbitrage principle is that the price P_0 is not unique and the spreads turn out to be unrealistically wide.

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FUNDAMENTAL THEOREMS OF ASSET PRICING

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DEFINITION (MARTINGALE)

A process $X = (X_t)_{t \in [0, T]}$ is a **MARTINGALE** if $\mathbf{E}[X_t | \mathcal{F}_s] = X_s$ for all $s \leq t$.

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- The argument for Efficient Market Hypothesis goes like this: So many speculators try to “beat the market” that all the (old) information is already incorporated in the prices. So, the new information must be independent of the past information.
- This means the Efficient Market Hypothesis is a paradox: It is true if and only if the speculators do not believe in it!

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DEFINITION (EQUIVALENCE OF PROBABILITY MEASURES)

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Having $\mathbf{P}[A] = 0$ does not make the event A impossible. So, the concept “possibility” has to be understood here in a vague sense.

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THEOREM (I FUNDAMENTAL THEOREM OF ASSET PRICING)

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REMARK (REPLICATION IN ARBITRAGE MODELS)

It is possible that an arbitrage model (no equivalent martingale measure $\tilde{\mathbf{P}}$) is complete.

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$$P_0 = \tilde{\mathbf{E}}[f(S_T)],$$

and more generally

$$P_t = \tilde{\mathbf{E}}[f(S_T)|\mathcal{F}_t].$$

This is why $\tilde{\mathbf{P}}$ is also called the **PRICING MEASURE**,

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- $V(\pi)$ of a self-financing π is always a martingale under $\tilde{\mathbf{P}}$.

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DEFINITION (BROWNIAN MOTION)

The **BROWNIAN MOTION**, or Wiener process, $W = (W_t)_{t \in [0, T]}$ is a stochastic process characterized by the following three properties:

- 1 the paths $t \mapsto W_t$ are **CONTINUOUS**,
- 2 the increments $W_{t+\Delta} - W_t$, $t \geq 0$, are **STATIONARY**, i.e. their probability laws are independent of t ,
- 3 the increments $W_{t_4} - W_{t_3}$, $W_{t_2} - W_{t_1}$, $t_1 < t_2 < t_3 < t_4$, are **INDEPENDENT**.

REMARK (PROPERTIES OF BROWNIAN MOTION)

■ GAUSSIANTY:

$$\mathbf{P}[W_t \in B] = \frac{1}{\sqrt{2\pi t}} \int_B e^{-\frac{x^2}{2t}} dx.$$

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THEOREM (ITÔ FORMULA)

Let $f(t, x) \in C^{1,2}([0, T] \times \mathbb{R})$. Then

$$df(t, W_t) = f_t(t, W_t)dt + f_x(t, W_t)dW_t + \frac{1}{2}f_{xx}(t, W_t)dt.$$

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The Itô formula follows from the second order Taylor formula (actually it **IS** the second order Taylor formula with $(dW_t)^2 = dt$).

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- From (3) we read that $(dS_t)^2 = \sigma^2 S_t^2 dt$. Thus,

$$df(t, S_t) = f_t(t, S_t)dt + f_x(t, S_t)dS_t + \frac{\sigma^2}{2} S_t^2 f_{xx}(t, S_t)dt.$$

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$$\begin{aligned} f(S_T) &= P_T = P(T, S_T) \\ &\stackrel{\text{Itô}}{=} P(0, S_0) + \int_0^T P_t(t, S_t) dt + \int_0^T P_x(t, S_t) dS_t \\ &\quad + \int_0^T \frac{\sigma^2}{2} S_t^2 P_{xx}(t, S_t) dt \end{aligned}$$

REPLICATION IN THE BLACK-SCHOLES MODEL

Let us calculate the price (and replicating portfolio) of an European vanilla option $f(S_T)$ in the Black-Scholes model.

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REPLICATION IN THE BLACK-SCHOLES MODEL

So, if $P(t, x)$ solves the **BLACK-SCHOLES BACKWARD PDE**

$$\begin{aligned}P_t(t, x) + \frac{\sigma^2}{2}x^2P_{xx}(t, x) &= 0, \\P(T, x) &= f(x),\end{aligned}$$

then $P_t = P(t, S_t)$ and $\gamma_t = P_x(t, S_t)$. In particular, the price of the option $f(S_T)$ is $P(0, S_0)$.

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REPLICATION IN THE BLACK-SCHOLES MODEL

Martingale Approach

Under \mathbf{P} we have

$$S_t = S_0 e^{\mu t + \sigma W_t - \frac{\sigma^2}{2} t},$$

where W is \mathbf{P} -Brownian motion.

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By **GIRSANOV THEOREM** the equivalent martingale measure $\tilde{\mathbf{P}}$ is unique and under $\tilde{\mathbf{P}}$ we have

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Next we “calculate” this conditional expectation.

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REPLICATION IN THE BLACK-SCHOLES MODEL

Finally we note that this unconditional expectation is easy to “calculate” by using the Gaussianity of the Brownian motion:

$$P_t = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f \left(S_t e^{\sigma\sqrt{T-t}y - \frac{\sigma^2}{2}(T-t)} \right) e^{-\frac{1}{2}y^2} dy. \quad (4)$$

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REMARK (QUADRATIC VARIATION DETERMINES THE PRICES)

The PDE Approach is valid whenever $(dS_t)^2 = \sigma^2 S_t^2 dt$. The Martingale Approach fails if the stock price is not driven by a Brownian motion (but because of the Feynman-Kac connection the result is true, nevertheless).

ACTS

- 1 TRADING STRATEGIES
- 2 PRICING PRINCIPLES: REPLICATION AND NO-ARBITRAGE
- 3 FUNDAMENTAL THEOREMS OF ASSET PRICING
- 4 REPLICATION IN THE BLACK-SCHOLES MODEL
- 5 WHY QUADRATIC VARIATION, OR BROWNIAN MOTION

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- Luckily the Brownian assumption was not needed in the PDE approach. What was essential was the existence of a non-trivial quadratic variation.

WHY QUADRATIC VARIATION, OR BROWNIAN MOTION

- Suppose the stock price process is continuous with $(dS_t)^2 = 0$ (this happens if S is differentiable). Then we have classical change-of-variables formula

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- The conclusion is that **STOCK PRICES MUST HAVE NON-TRIVIAL QUADRATIC VARIATION.**

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- The curtain falls -
Any questions, my beloved audience?