What is the price of the future?

An Unfinished Opera in Five Acts

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DISCLAIMER

In this talk I try to follow the EINSTEIN'S MAXIM:

> *Things should be made as simple as possible, but not simpler.*

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This means that, as far as technical details are concerned, I will cheat!

Overture

- Consider a Toy Model: Today's stock price S_0 is 100 ISK. Tomorrow's stock price S_1 is 200 ISK with probability 90% and 90 ISK with probability 10%:

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$$

- Mr. K. offers a European Call-Option: We get a right to buy tomorrow the stock S with today's price 100 ISK: Formula for our profit is

$$f(S_1) = (S_1 - 100)^+,$$

where $x^+ := \max(x, 0)$.

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- But if we pay 90 ISK, Mr. K. can do a follows: He takes a bank loan of 10 ISK and buys one stock (he already got 90 ISK by selling the call-option). Now, if the stock price will go up, Mr. K. gives us the stock in return of 100 ISK. After paying his bank loan Mr. K. has made a profit of 90 ISK. If, on the other hand, the stock will go down we will not exercise our option. Now Mr. K. sells his stock in the market and after paying his bank loan he has made a profit of 80 ISK.

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- By the way: The correct price in this case is 100/11 ISK. If the price is higher then Mr. K. can generate arbitrage; if the price is lower then we, the buyer, can generate arbitrage.

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- By the way: The correct price in this case is 100/11 ISK. If the price is higher then Mr. K. can generate arbitrage; if the price is lower then we, the buyer, can generate arbitrage.
- Actually, the arbitrage-free price is independent of the probabilities of stock going up or down. This should be clear from the fact that Mr. K.'s arbitrage strategy was independent of probabilities.

Acts

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Trading Strategies

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- The Bond, or bank account, $B = (B_t)_{t \in [0,T]}$ is the riskless asset. We assume that we live in the discounted world: $B_t = 1$ for all t.
- The Stock $S = (S_t)_{t \in [0,T]}$ is the risky asset. It is random: The prices S_t are not known before time t.

Trading Strategies

Definition (Trading Strategy)

Trading Strategy is $(\pi_t)_{t\in[0,T]} = (\beta_t, \gamma_t)_{t\in[0,T]}$. Here β_t tells the amount of bonds and γ_t the amount of stocks the investor has in her portfolio at time t. The amounts β_t and γ_t can depend on the price process S up to time t, but not on the future prices of S.

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Definition (Value of a Self-Financing Strategy)

The Value of a trading strategy π at time t is $V_t(\pi) = \beta_t + \gamma_t S_t$. Trading strategy is Self-Financing if

$$\mathrm{d}V_t(\pi) = \gamma_t \mathrm{d}S_t. \tag{1}$$

Trading Strategies

Remark (Self-financing and Budget Constraints)

- In (1) the differentials are understood limits in the forward sense: $\mathrm{d}S_t \approx S_{t+\Delta} - S_t$ when $\Delta > 0$ is small.

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- In (1) the differentials are understood limits in the forward sense: $\mathrm{d}S_t \approx S_{t+\Delta} - S_t$ when $\Delta > 0$ is small.
- Equation (1) is actually a budget constraint. Indeed, consider it in discrete time points $t < t+1$. Then says

$$V_{t+1}(\pi) - V_t(\pi) = \gamma_t \left(S_{t+1} - S_t\right),$$

which is actually equivalent to

$$\beta_t + \gamma_t S_t = \beta_{t+1} + \gamma_{t+1} S_t.$$

This means that the value of the portfolio remain unchanged when the portfolio if rebalanced and all the changes in the value come from the changes of S.

Acts

Pricing Principles: Replication and No-Arbitrage

Definition (Option)

An Option is simply a function of the underlying stock. We consider here only European Vanilla Options, i.e. options that are of the form $f(S_T)$, where $f : \mathbb{R} \to \mathbb{R}$ is some function.

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Definition (Replication Principle)

The Replication Price of $f(S_T)$ is the initial capital $V_0(\pi)$ needed to construct a self-financing trading strategy π for which

$$V_T(\pi) = f(S_T).$$

Here π is the Replicating Portfolio.

Pricing Principles: Replication and No-Arbitrage

Remark (Replication)

Suppose we can write

$$f(S_T) = c + \int_0^T \xi_t \, \mathrm{d}S_t. \tag{2}$$

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Then c is the price of the option and the replicating portfolio π is determined by: $\gamma_t = \xi_t$, $\beta_0 + \gamma_0 S_0 = c$, and β_t is determined from these and the fact that the portfolio is self-financing.

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The key question is: Can one construct replicating portfolios, i.e. representation of type (2) (in theory, in practise)?

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Let $P = (P_t)_{t\in[0,T]}$ be a price process for the option $f(S_T)$, i.e. P_t is the price of the option $f(S_T)$ at time t. (Obviously $P_T = f(S_T)$, but it is P_0 that we are interested in.)

Pricing Principles: Replication and No-Arbitrage

Definition (No-Arbitrage Principle)

P_0 is an arbitrage-free price for $f(S_T)$ if there is no such trading strategy $\pi = (\beta, \gamma, \delta)$ satisfying the self-financing condition

$$\mathrm{d}V_t(\pi) = \gamma_t \mathrm{d}S_t + \delta_t \mathrm{d}P_t$$

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Remark (Arbitrage Spread)

The problem with the no-arbitrage principle is that the price P_0 is not unique and the spreads turn out to be unrealistically wide.

Acts

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Definition (Martingale)

A process $X = (X_t)_{t\in[0,T]}$ is a Martingale if $\mathbf{E}[X_t|\mathcal{F}_s] = X_s$ for all $s \leq t$.

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- The argument for Efficient Market Hypothesis goes like this: So many speculators try to "beat the market" that all the (old) information is already incorporated in the prices. So, the new information must be independent of the past information.
- This means the Efficient Market Hypothesis is a paradox: It is true if and only if the speculators do not believe in it!

Fundamental Theorems of Asset Pricing

In the overture we saw how to make arbitrage (if the price was not right) without probabilities. Actually, the existence of arbitrage possibilities is independent of probability up-to possibility: The definition of arbitrage speaks about the probability $\mathbf{P}$ only at the "zero-one level".

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Definition (Equivalence of Probability Measures)

A probability measure $\tilde{\mathbf{P}}$ is equivalent to the probability measure $\mathbf{P}$ if they have same zero-sets: $\tilde{\mathbf{P}}[A] = 0$ if and only if $\mathbf{P}[A] = 0$.

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Having $\mathbf{P}[A] = 0$ does not make the event A impossible. So, the concept "possibility" has to be understood here in a vague sense.

Fundamental Theorems of Asset Pricing

Theorem (I Fundamental Theorem of Asset Pricing)

A market model $(S, \mathbf{P})$ *is free of arbitrage if and only if there exists* $\tilde{\mathbf{P}}$ *equivalent to* $\mathbf{P}$ *such that* S *is martingale with respect to* $\tilde{\mathbf{P}}$*.*

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Remark (Replication in Arbitrage Models)

It is possible that an arbitrage model (no equivalent martingale measure $\tilde{\mathbf{P}}$) is complete.

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- Arbitrage-free price of an option $f(S_T)$ is its expected value under an equivalent martingale measure:

$$P_0 = \tilde{\mathbf{E}}[f(S_T)],$$

and more generally

$$P_t = \tilde{\mathbf{E}}[f(S_T)|\mathcal{F}_t].$$

This is why $\tilde{\mathbf{P}}$ is also called the Pricing Measure,

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  This is why $\tilde{\mathbf{P}}$ is also called the Pricing Measure,
- $V(\pi)$ of a self-financing π is always a martingale under $\tilde{\mathbf{P}}$.

Acts

Replication in the Black-Scholes Model

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Definition (Brownian Motion)

The Brownian Motion, or Wiener process, $W = (W_t)_{t\in[0,T]}$ is a stochastic process characterized by the following three properties:

1. the paths $t \mapsto W_t$ are continuous,
2. the increments $W_{t+\Delta} - W_t$, $t \geq 0$, are stationary, i.e. their probability laws are independent of t,
3. the increments $W_{t_4} - W_{t_3}$, $W_{t_2} - W_{t_1}$, $t_1 < t_2 < t_3 < t_4$, are independent.

Replication in the Black-Scholes Model

Remark (Properties of Brownian Motion)

- Gaussianity:

$$\mathbf{P}[W_t \in B] = \frac{1}{\sqrt{2\pi t}} \int_B e^{-\frac{x^2}{2t}} \, \mathrm{d}x.$$

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- Quadratic variation:

$$(\mathrm{d}W_t)^2 = \mathrm{d}t.$$

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Theorem (Itô Formula)

Let $f(t,x) \in C^{1,2}([0,T] \times \mathbb{R})$. Then

$$\mathrm{d}f(t,W_t) \;=\; f_t(t,W_t)\mathrm{d}t + f_x(t,W_t)\mathrm{d}W_t + \frac{1}{2} f_{xx}(t,W_t)\mathrm{d}t.$$

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The Itô formula follows from the second order Taylor formula (actually it IS the second order Taylor formula with $(\mathrm{d}W_t)^2 = \mathrm{d}t$).

Replication in the Black-Scholes Model

Definition (Black-Scholes Model)

In the Black-Scholes Model the dynamics of the stock price are given by the stochastic differential equation (SDE)

$$\frac{\mathrm{d}S_t}{S_t} = \mu \mathrm{d}t + \sigma \mathrm{d}W_t. \quad (3)$$

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- From (3) we read that $(\mathrm{d}S_t)^2 = \sigma^2 S_t^2 \mathrm{d}t$. Thus,

$$\mathrm{d}f(t, S_t) = f_t(t, S_t)\mathrm{d}t + f_x(t, S_t)\mathrm{d}S_t + \frac{\sigma^2}{2} S_t^2 f_{xx}(t, S_t)\mathrm{d}t.$$

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PDE Approach

Ansatz: $P_t = P(t, S_t)$ and $\gamma_t = P_x(t, S_t)$. Then

$$f(S_T) \quad = \quad P_T \quad = \quad P(T, S_T)$$

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$$\begin{aligned} f(S_T) &= P_T = P(T, S_T) \\ &\overset{\text{Itô}}{=} P(0, S_0) + \int_0^T P_t(t, S_t)\,\mathrm{d}t + \int_0^T P_x(t, S_t)\,\mathrm{d}S_t \\ &\quad + \int_0^T \frac{\sigma^2}{2} S_t^2 P_{xx}(t, S_t)\,\mathrm{d}t \end{aligned}$$

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Replication in the Black-Scholes Model

So, if $P(t,x)$ solves the Black-Scholes Backward PDE

$$\begin{aligned} P_t(t,x) + \frac{\sigma^2}{2} x^2 P_{xx}(t,x) &= 0, \\ P(T,x) &= f(x), \end{aligned}$$

then $P_t = P(t,S_t)$ and $\gamma_t = P_x(t,S_t)$. In particular, the price of the option $f(S_T)$ is $P(0,S_0)$.

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then $P_t = P(t,S_t)$ and $\gamma_t = P_x(t,S_t)$. In particular, the price of the option $f(S_T)$ is $P(0,S_0)$.

- Since γ_t is the "spot derivative" P_x of the value function P, this replication, or hedge, is called the Δ-hedge.

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So, if $P(t,x)$ solves the Black-Scholes Backward PDE

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then $P_t = P(t,S_t)$ and $\gamma_t = P_x(t,S_t)$. In particular, the price of the option $f(S_T)$ is $P(0,S_0)$.

- Since γ_t is the "spot derivative" P_x of the value function P, this replication, or hedge, is called the Δ-hedge.
- The Black-Scholes Backward PDE can be solved numerically, but rarely analytically.

Replication in the Black-Scholes Model

Martingale Approach

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$$S_t = S_0 e^{\mu t + \sigma W_t - \frac{\sigma^2}{2} t},$$

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Next we “calculate” this conditional expectation.

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Finally we note that this unconditional expectation is easy to "calculate" by using the Gaussianity of the Brownian motion:

$$P_t = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f\left(S_t e^{\sigma\sqrt{T-t}y - \frac{\sigma^2}{2}(T-t)}\right) e^{-\frac{1}{2}y^2}\,\mathrm{d}y. \tag{4}$$

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Remark (Quadratic Variation Determines the Prices)

The PDE Approach is valid whenever $(\mathrm{d}S_t)^2 = \sigma^2 S_t^2 \mathrm{d}t$. The Martingale Approach fails if the stock price is not driven by a Brownian motion (but because of the Feynman-Kac connection the result is true, nevertheless).

Acts

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- Actual stock data does not support this assumption: There is overwhelming evidence the log-prices are neither independent nor Gaussian (never mind the Central Limit Theorem).
- Luckily the Brownian assumption was not needed in the PDE approach. What was essential was the existence of a non-trivial quadratic variation.

Why Quadratic Variation, or Brownian Motion

- Suppose the stock price process is continuous with $(\mathrm{d}S_t)^2 = 0$ (this happens if S is differentiable). Then we have classical change-of-variables formula

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- The conclusion is that STOCK PRICES MUST HAVE NON-TRIVIAL QUADRATIC VARIATION.

Finale

- The story we have explained here is very modern (even when it comes to its mathematical foundations), and in many ways an unwritten opera.

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- The curtain falls -

Any questions, my beloved audience?