Prediction law of fractional Brownian motion

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References

- Sottinen, T. and Viitasaari, L. (2017)
  Prediction Law of Fractional Brownian Motion.
  Statistics & Probability Letters **129**, 155–166.

- Shokrollahi, F. and Sottinen, T. (2017)
  Hedging in fractional Black–Scholes model with transaction costs.
  Statistics & Probability Letters **130**, 85–91.

- Sottinen, T. and Viitasaari, L. (2018)
  Conditional-Mean Hedging Under Transaction Costs in Gaussian Models.
  International Journal of Theoretical and Applied Finance **21**, no. 2.

Outline

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Fractional Brownian motion

The fractional Brownian motion $B^H = (B_t^H)_{t \geq 0}$ with Hurst index $H \in (0, 1)$ is the (upto a multiplicative constant) unique centered Gaussian process that has stationary increments and is H-self-similar, i.e,

$$(B_{at}^H)_{t \geq 0} \stackrel{d}{=} (a^H B_t^H)_{t \geq 0}.$$

The covariance function of B^H is

$$\begin{aligned} r_H(t, s) &:= \mathbb{C}\mathrm{ov}\left[B_t^H, B_s^H\right] \\ &= \frac{1}{2}\left[t^{2H} + s^{2H} - |t - s|^{2H}\right]. \end{aligned}$$

For $H = \frac{1}{2}$, $B^{\frac{1}{2}} = W$, the standard Brownian motion.

Fractional Brownian motion

The fractional Brownian motion is an invertible Gaussian Volterra process, i.e., there exists a Brownian motion W that is constructed from the fractional Brownian motion B^H in an non-anticipative way such that the fractional Brownian motion can be recovered from the Brownian motion in the non-anticipative way

$$B_t^H = \int_0^t k_H(t,s)\,\mathrm{d}W_s.$$

The kernel k_H is known explicitly, but we do not bother to show its ugly form here.

NB: We have $\mathcal{F}_t^{B^H} = \mathcal{F}_t^W$.

Outline

Gaussian prediction in general

Let X and Y be Hilbert-valued Gaussian things. Then we can decompose

$$X = P_Y X + (I - P_Y)X,$$

where $P_Y X = \mathbb{E}[X|Y]$ and $(I - P_Y)X$ is independent of Y due to its orthogonality.

It follows that the conditional thing $X|Y$ is Gaussian with Y-measurable mean $\mathbb{E}[X|Y]$ and deterministic covariance

$$\begin{aligned} \mathbb{V}\mathrm{ar}[X|Y] &= \mathbb{V}\mathrm{ar}\left[(I - P_Y)X \mid Y\right] \\ &= \mathbb{V}\mathrm{ar}\left[(I - P_Y)X\right] \end{aligned}$$

Outline

Prediction of fractional Brownian motion

We want to know the law of the conditional process

$$\hat{B}_t(u) \;=\; B_t^H | \mathcal{F}_u^{B^H}, \qquad t \geq u.$$

We know that $\hat{B}^H(u)$ is Gaussian with $\mathcal{F}_u^{B^H}$-measurable mean and deterministic covariance function. Thus, it only remains to find out the conditional mean and covariance. In finding those we only use the following two facts:

$$B_t^H \;=\; \int_0^t k_H(t,s)\,\mathrm{d}W_s, \tag{1}$$

$$\mathcal{F}_u^{B^H} \;=\; \mathcal{F}_u^W. \tag{2}$$

(And the fact that we know explicitly how W is constructed from B^H.)

Prediction of fractional Brownian motion

The conditional mean is

$$\begin{aligned}
\hat{m}_t^H(u) &:= \mathbb{E}\left[B_t^H \middle| \mathcal{F}_u^{B^H}\right] \\
&= \mathbb{E}\left[\int_0^t k_H(t,s)\,\mathrm{d}W_s \,\middle|\, \mathcal{F}_u^{B^H}\right] \\
&= \mathbb{E}\left[\int_0^t k_H(t,s)\,\mathrm{d}W_s \,\middle|\, \mathcal{F}_u^{W}\right] \\
&= \mathbb{E}\left[\int_0^u k_H(t,s)\,\mathrm{d}W_s + \int_u^t k_H(t,s)\,\mathrm{d}W_s \,\middle|\, \mathcal{F}_u^{W}\right] \\
&= \int_0^u k_H(t,s)\,\mathrm{d}W_s + \mathbb{E}\left[\int_u^t k_H(t,s)\,\mathrm{d}W_s\right] \\
&= \int_0^u k_H(t,s)\,\mathrm{d}W_s.
\end{aligned}$$

Prediction of fractional Brownian motion

For the conditional variance, we first note that

$$\begin{aligned} B_t^H - \hat{m}_t^H(u) &= \int_0^t k_H(t,s)\,\mathrm{d}W_s - \int_0^u k_H(t,s)\,\mathrm{d}W_s \\ &= \int_u^t k_H(t,s)\,\mathrm{d}W_s. \end{aligned}$$

Thus,

$$\begin{aligned} \hat{r}_H(t,s|u) &:= \mathbb{C}\mathrm{ov}\left[B_t^H, B_s^H \middle| \mathcal{F}_u^{B^H}\right] \\ &= \mathbb{E}\left[\int_u^t k_H(t,v)\,\mathrm{d}W_v \int_u^s k_H(s,w)\,\mathrm{d}W_w \,\middle|\, \mathcal{F}_u^{B^H}\right] \\ &= \mathbb{E}\left[\int_u^t k_H(t,v)\,\mathrm{d}W_v \int_u^s k_H(s,w)\,\mathrm{d}W_w \,\middle|\, \mathcal{F}_u^{W}\right] \\ &= \mathbb{E}\left[\int_u^t k_H(t,v)\,\mathrm{d}W_v \int_u^s k_H(s,w)\,\mathrm{d}W_w\right]. \end{aligned}$$

Prediction of fractional Brownian motion

By using the Itô isometry, we obtain

$$\hat{r}_H(t,s|u) \;=\; \int_u^{\min(t,s)} k_H(t,v)k_H(s,v)\,\mathrm{d}v.$$

We also note that

$$r_H(t,s) \;=\; \int_0^{\min(t,s)} k_H(t,v)k_H(s,v)\,\mathrm{d}v.$$

Consequently,

$$\hat{r}_H(t,s|u) \;=\; r_H(t,s) - \int_0^{u} k_H(t,v)k_H(s,v)\,\mathrm{d}v.$$

Prediction of fractional Brownian motion

What we have done, is the proof of the following theorem:

Theorem

The conditional fractional Brownian motion $\hat{B}^H(u) = B^H | \mathcal{F}_u^{B^H}$ is a Gaussian process with mean and covariance

$$\begin{aligned} \hat{m}_t^H(u) &= \int_0^u k_H(t,s)\,\mathrm{d}W_s, \\ \hat{r}_H(t,s|u) &= r_H(t,s) - \int_0^u k_H(t,v)k_H(s,v)\,\mathrm{d}v. \end{aligned}$$

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I Thank You for Your time!

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I Thank You for Your time!
Any Questions?