Necessary and Sufficient Conditions for Hölder Continuity of Gaussian Processes

Tommi Sottinen[1]

University of Vaasa, Finland

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[1] http://www.uva.fi/∼tsottine/

Abstract

We prove the following simple necessary and sufficient condition for the Hölder continuity of Gaussian processes:

Theorem (Azmoodeh-Sottinen-Viitasaari-Yazigi)

A centered Gaussian process $X = (X_t)_{t \in [0,T]}$ is Hölder continuous with parameter H IF AND ONLY IF, *for all $\varepsilon > 0$,*

$$E\left[(X_t - X_s)^2\right] \quad \leq \quad c_\varepsilon |t-s|^{2H-\varepsilon}$$

The talk in based on the article

Azmoodeh, E., Sottinen, T., Viitasaari, L. and **Yazigi, A.** (2014) Necessary and Sufficient Conditions for Hölder Continuity of Gaussian Processes. *Statistics & Probability Letters*, to appear.

Notation

In what follows X will always be a centered Gaussian process on the interval $[0, T]$.

For a centered Gaussian family $\xi = (\xi_\tau)_{\tau \in \mathbb{T}}$ we denote

$$\begin{aligned} d_\xi^2(\tau, \tau') &:= \mathbb{E}[(\xi_\tau - \xi_{\tau'})^2], \\ \sigma_\xi^2(\tau) &:= \mathbb{E}[\xi_\tau^2]. \end{aligned}$$

Theorem (Fernique 1964)

Assume that for some positive ε, *and* $0 \leq s \leq t \leq \varepsilon$, *there exists a nondecreasing function* Ψ *on* $[0, \varepsilon]$ *such that* $d_X^2(s,t) \leq \Psi^2(t-s)$ *and*

$$\int_0^\varepsilon \frac{\Psi(u)}{u\sqrt{\log u}}\,\mathrm{d}u \;<\; \infty.$$

Then X *is continuous.*

The convergence of the integral is NOT NECESSARY for continuity!

THEOREM (DUDLEY 1967)

Let $N(\varepsilon) := N([0,T], d_X, \varepsilon)$ *denote the minimum number of closed balls of radius* ε *in the (pseudo) metric* d_X *needed to cover* $[0,T]$. *If*

$$\int_0^\infty \sqrt{\log N(\varepsilon)}\,\mathrm{d}\varepsilon \;<\; \infty,$$

then X *is continuous.*

Like in the case of the Fernique's condition, the finiteness of the Dudley integral above is NOT NECESSARY for continuity. However, for STATIONARY PROCESSES IT IS NECESSARY AND SUFFICIENT.

Let $B_{d_X}(t,\varepsilon)$ a ball with radius ε at center t in the (pseudo) metric d_X. A probability measure μ on $([0,T],d_X)$ is called a MAJORIZING MEASURE if

$$\sup_{t\in[0,T]}\int_0^\infty \sqrt{\log\frac{1}{\mu\left(B_{d_X}(t,\varepsilon)\right)}}\,\mathrm{d}\varepsilon<\infty.$$

Theorem (Talagrand 1987)

The Gaussian process X is continuous IF AND ONLY IF *there exists a majorizing measure μ on $([0,T],d_X)$ such that*

$$\lim_{\delta\to 0}\sup_{t\in[0,T]}\int_0^\delta \sqrt{\log\frac{1}{\mu(B_{d_X}(t,\varepsilon))}}\,\mathrm{d}\varepsilon=0.$$

Theorem (Azmoodeh-Sottinen-Viitasaari-Yazigi)

A centered Gaussian process $X = (X_t)_{t\in[0,T]}$ is Hölder continuous with parameter H IF AND ONLY IF*, for all $\varepsilon > 0$,*

$$d_X(t,s) \quad \le \quad c_\varepsilon |t-s|^{H-\varepsilon}$$

Remark (Relation to Kolmogorov–Čentsov)

- If part follows from KC and existence of all moments. Otherwise Gaussianity is not essential.
- Only if part uses Gaussianity.

Remark (ε-gap)

The ε-gap in the Theorem ASVY cannot be removed. Indeed, let

$$X_t = (\log \log 1/t)^{-1/2} B_t,$$

where B is an H-fractional Brownian motion. Then, by the Law of Iterated Logarithm, X is Hölder continuous with any index $a < H$, but the condition in Theorem ASVY does not hold without ε.

The if part of theorem ASVY follows from KC. The only if part follows from the following elementary lemma:

Lemma

If $\sup_{\tau \in \mathbb{T}} |\xi_\tau| < \infty$ *then* $\sup_{\tau \in \mathbb{T}} \sigma_\xi^2(t) < \infty$.

Proof of Lemma.

Since $\sup_{\tau\in\mathbb{T}}|\xi_\tau| < \infty$, $\mathbb{P}[\sup_{\tau\in\mathbb{T}}|\xi_\tau| < x] > 0$ for a large enough $x \in \mathbb{R}$. Now, for all $\tau \in \mathbb{T}$, we have that

$$\begin{aligned}
\mathbb{P}\left[\sup_{\tau\in\mathbb{T}}|\xi_\tau| < x\right] &\leq \mathbb{P}\left[|\xi_\tau| < x\right] = \mathbb{P}\left[\left|\frac{\xi_\tau}{\sigma_\xi(\tau)}\right| < \frac{x}{\sigma_\xi(\tau)}\right] \\
&= \frac{2}{\sqrt{2\pi}}\int_0^{x/\sigma_\xi(\tau)} e^{-\frac{1}{2}z^2}\,\mathrm{d}z \\
&\leq \frac{2}{\sqrt{2\pi}}\frac{x}{\sigma_\xi(\tau)}
\end{aligned}$$

$$\Rightarrow \qquad \sigma_\xi^2(\tau) \leq \frac{2x^2}{\pi\mathbb{P}\left[\sup_{\tau\in\mathbb{T}}|\xi_\tau| < x\right]^2},$$

and the claim follows from this. □

Proof of Only If.

Suppose X is Hölder continuous of order $a = H - \varepsilon$. Define a Gaussian family

$$\xi_{t,s} = \frac{X_t - X_s}{|t - s|^{H-\varepsilon}}.$$

Since ξ is a bounded centered Gaussian family, we obtain by the previous Lemma that $\sup_{(t,s)\in[0,T]^2} \sigma_\xi^2(t,s) < \infty$. This means that

$$d_X(t,s) \leq c_\varepsilon |t - s|^{H-\varepsilon}.$$

□

Corollary (Stationary Increments)

X is Hölder continuous of any order $a < H$ if and only if

$$\sigma_X^2(t) \leq c_\varepsilon t^{2H-\varepsilon}, \quad \textit{for all } \varepsilon > 0.$$

Corollary (Stationary)

X with spectral measure Δ is Hölder continuous of any order $a < H$ if and only if

$$\int_0^\infty \big(1 - \cos(\lambda t)\big)\, \Delta(\mathrm{d}\lambda) \leq c_\varepsilon t^{2H-\varepsilon} \quad \textit{for all } \varepsilon > 0.$$

A bounded process can be viewed as an $L^2([0,T])$-valued random variable. Hence, the covariance operator admits a square root with kernel K, and we may represent X as a Gaussian Fredholm process:

$$X_t = \int_0^T K(t,s)\,\mathrm{d}W_s,$$

where W is a Brownian motion and $K \in L^2([0,T]^2)$.

Corollary (Fredholm)

X is Hölder continuous of any order $a < H$ if and only if

$$\int_0^T |K(t,u) - K(s,u)|^2\,\mathrm{d}u \le c_\varepsilon |t-s|^{2H-\varepsilon} \quad \textit{for all } \varepsilon > 0.$$

A process X is *self-similar* with index $\beta > 0$ if

$$(X_{at})_{0 \le t \le T/a} \overset{d}{=} (a^{\beta} X_t)_{0 \le t \le T}, \quad \text{for all } a > 0.$$

In the Gaussian case this means that

$$d_X(t,s) = a^{-\beta} d_X(at, as) \quad \text{for all } a > 0.$$

So, it is clear that X cannot be Hölder continuous of order $H > \beta$.

Let $\mathcal{H}_t^X$ be the closed linear subspace of $L^2(\Omega)$ generated by the Gaussian random variables $\{X_s; s \le t\}$ and $\mathcal{H}_{0+}^X := \cap_{t \in (0,T]} \mathcal{H}_t^X$.

X is PURELY NON-DETERMINISTIC (PND) if $\mathcal{H}_{0+}^X$ is trivial.

Theorem (Yazigi 2014)

PND Gaussian self-similar process admits the representation

$$X_t = \int_0^t t^{\beta-\frac{1}{2}} F\left(\frac{u}{t}\right) \mathrm{d}W_u, \quad F \in L^2([0,1]).$$

Corollary (Self-Similar)

Let X be PND self-similar with index β. Then X is Hölder continuous of any order $a < H$ if and only if

1. $\int_s^t t^{2\beta-1} F(\frac{u}{t})^2 \,\mathrm{d}u \le c_\varepsilon |t-s|^{2H-\varepsilon}$,
2. $\int_0^s \left| t^{\beta-\frac{1}{2}} F\left(\frac{u}{t}\right) - s^{\beta-\frac{1}{2}} F\left(\frac{u}{s}\right)\right|^2 \mathrm{d}u \le c_\varepsilon |t-s|^{2H-\varepsilon}$

References

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Thank You for listening!

Questions or comments will be most appreciated!