Gaussian Fredholm Processes

with Applications

Tommi Sottinen

University of Vaasa, Finland

(Based on a joint work with Lauri Viitasaari, Aalto University, Finland)

Yu.V.Linnik Centennial Conference, St. Petersburg, Russia
September 14-18, 2015

Abstract

Motto: Gaussian processes are difficult, Brownian motion is easy.

We show that every separable Gaussian process with integrable variance function admits a Fredholm representation with respect to a Brownian motion.

We extend the Fredholm representation to a transfer principle and develop stochastic analysis by using it. In particular, we prove an Itô formula that is, as far as we know, the most general Skorohod-type Itô formula for Gaussian processes so far.

Finally, we give applications to equivalence in law and series expansions of Gaussian processes.

Outline

1 Fredholm Representation

2 Transfer Principle

3 Applications

Outline

1 Fredholm Representation

2 Transfer Principle

3 Applications

Theorem (Fredholm Representation)

Let $X = (X_t)_{t\in[0,T]}$ be a separable centered Gaussian process. Then there exists a kernel $K_T \in L^2([0,T]^2)$ and a Brownian motion $W = (W_t)_{t\geq 0}$, independent of T, such that

$$X_t = \int_0^T K_T(t,s)\,\mathrm{d}W_s$$

if and only if the covariance R of X satisfies the trace condition

$$\int_0^T R(t,t)\,\mathrm{d}t < \infty.$$

- The Fredholm Kernel K_T usually depends on T even if R does not.
- K_T may be assumed to be symmetric.
- K_T is unique in the sense that if there is another representation with kernel $\tilde{K}_T$, then $\tilde{K}_T = UK_T$ for some unitary operator U on $L^2([0, T])$.
- The Fredholm Representation Theorem holds also for the parameter space $\mathbb{R}_+$, but the trace condition seldom holds, i.e. typically
$$\int_0^\infty R(t,t)\,\mathrm{d}t = \infty.$$
- If the covariance R is degenerate, one needs to extend the probability space to carry the Brownian motion.

- K_T (operator) can be constructed from R_T (operator) as the unique positive symmetric square-root, i.e. the operator K_T is a limit of polynomials:

$$K_T = \lim_{n\to\infty} P_n(R_T).$$

- The positive symmetric square-root is different from the Cholesky square-root. Indeed, the Cholesky square-root would correspond the Volterra representation

$$X_t = \int_0^t K(t,s)\,\mathrm{d}W_s.$$

  The Volterra representation does not hold for Gaussian processes in general.

Consider a truncated series expansion

$$X_t = \sum_{k=1}^{n} e_k^T(t)\xi_k,$$

where ξ_k are independent standard normal random variables and $e_k^T(t) = \int_0^t \tilde{e}_k^T(s)\,\mathrm{d}s$, where $\tilde{e}_k^T$, $k \in \mathbb{N}$, is an orthonormal basis in $L^2([0,T])$. This process is not *purely non-deterministic* and consequently, X cannot have Volterra representation while X admits a Fredholm representation. On the other hand, by choosing $\tilde{e}_k^T$ to be the trigonometric basis on $L^2([0,T])$, X is a finite-rank approximation of the Karhunen–Loève representation of standard Brownian motion on $[0,T]$. Hence by letting $n \to \infty$ we obtain a standard Brownian motion, and hence a Volterra process.

Let W be a standard Brownian motion and consider the Brownian bridge B. The orthogonal representation of B is

$$B_t = W_t - \frac{t}{T} W_T.$$

Consequently, B has a Fredholm representation with kernel

$$K_T(t,s) = \mathbf{1}_{[0,t)}(s) - \frac{t}{T}.$$

The canonical representation of the Brownian bridge is

$$B_t = (T-t) \int_0^t \frac{1}{T-s} \, \mathrm{d}W_s.$$

Hence B has also a Volterra representation with kernel

$$K(t,s) = \frac{T-t}{T-s}.$$

By the Mercer's theorem

$$R(t,s) = \sum_{i=1}^{\infty} \lambda_i^T e_i^T(t) e_i^T(s),$$

where $(\lambda_i^T)_{i=1}^{\infty}$ and $(e_i^T)_{i=1}^{\infty}$ are the eigenvalues and the eigenfunctions of the covariance operator

$$R_T f(t) = \int_0^T f(s) R(t,s) \, \mathrm{d}s.$$

Moreover, $(e_i^T)_{i=1}^{\infty}$ is an orthonormal system on $L^2([0,T])$.

Since R_T is a covariance-operator, it admits a square-root operator K_T. By the trace condition R_T is trace-class, and hence K_T is Hilbert-Schmidt. Thus, K_T admits a Kernel.

Indeed,

$$K_T(t,s) = \sum_{i=1}^{\infty} \sqrt{\lambda_i^T} e_i^T(t) e_i^T(s).$$

Now K_T is obviously symmetric and we have

$$R(t,s) = \int_0^T K_T(t,u) K_T(s,u) \, \mathrm{d}u$$

from which the Fredholm Representation follows by enlarging the probability space, if needed. $\square$

Outline

1 Fredholm Representation

2 Transfer Principle

3 Applications

The adjoint operator Γ^* of a kernel $\Gamma \in L^2([0, T]^2)$ is defined by linearly extending the relation

$$\Gamma^* \mathbf{1}_{[0,t)} = \Gamma(t, \cdot).$$

Remark

If $\Gamma(\cdot, s)$ is of bounded variation for all s and f is nice enough, then

$$\Gamma^* f(s) = \int_0^T f(t)\, \Gamma(\mathrm{d}t, s).$$

Theorem (Transfer Principle)

Let X be Gaussian Fredholm process with kernel K_T. Let D_T, δ_T, D_T^W and δ_T^W be the Malliavin derivative and the Skorohod integral with respect to X and to the Brownian motion W. Then

$$\delta_T = \delta_T^W K_T^* \quad \textit{and} \quad K_T^* D_T = D_T^W.$$

Proof: Trivial. □

Theorem (Itô Formula)

Let X be centered Gaussian process with covariance R and let $f \in C^2$. Then

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\,\delta X_s + \frac{1}{2}\int_0^t f''(X_s)\,\mathrm{d}R(s,s),$$

if anything.

Outline

1. Fredholm Representation
2. Transfer Principle
3. Applications

Let us show how to use the Fredholm Representation and the Transfer Principle to analyze equivalence of Gaussian laws.

Recall the Hitsuda Representation Theorem: A centered Gaussian process $\tilde{W}$ is equivalent to a Brownian motion W if and only if there exists a Volterra kernel $\ell \in L^2([0,T]^2)$ such that

$$\mathrm{d}\tilde{W}_t = \mathrm{d}W_t + \int_0^t \ell(t,s)\,\mathrm{d}W_s \cdot \mathrm{d}t.$$

Now, let $\tilde{X}$ and X be Gaussian Fredholm processes with

$$\begin{aligned}\tilde{X}_t &= \int_0^T \tilde{K}_T(t,s)\,\mathrm{d}W_s, \\ X_t &= \int_0^T K_T(t,s)\,\mathrm{d}W_s.\end{aligned}$$

Suppose then that $\tilde{X}$ has (also) representation

$$\tilde{X}_t = \int_0^T K_T(t,s)\,\mathrm{d}\tilde{W}_s$$

where $\tilde{W}$ and W are equivalent.

Then, obviously $\tilde{X}$ and X are equivalent. By plugging in the Hitsuda connection we obtain

$$\tilde{X}_t = \int_0^T \left[K_T(t,s) + \int_s^T K_T(t,u)\ell(u,s)\,\mathrm{d}u \right] \mathrm{d}W_s.$$

Thus, we have shown the following:

Theorem (Equivalence of Laws)

Let X and $\tilde{X}$ be two Gaussian process with Fredholm kernels K_T and $\tilde{K}_T$, respectively. If there exists a Volterra kernel $\ell \in L^2([0,T]^2)$ such that

$$\tilde{K}_T(t,s) = K_T(t,s) + \int_s^T K_T(t,u)\ell(u,s)\,\mathrm{d}u,$$

then X and $\tilde{X}$ are equivalent in law.

If the kernel K_T satisfies appropriate non-degeneracy property, then the condition above is also necessary.

In the same way, as in the case of equivalence of laws, we see that:

Theorem (Series representation)

Let X be a Gaussian Fredholm process with kernel K_T and let φ_k^T, $k \in \mathbb{N}$, be any orthonormal basis in $L^2([0,T])$. Then

$$X_t = \sum_{k=1}^{\infty} \int_0^T K_T(t,s)\varphi_k^T(s)\,\mathrm{d}s \cdot \xi_k,$$

where ξ_k, $k \in \mathbb{N}$, are i.i.d. standard Gaussian random variables.

The series above converges in $L^2(\Omega)$; and also almost surely uniformly if and only if X is continuous.

Thank you for listening!
Any questions?