Gaussian Fredholm Processes

with Applications

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Abstract

Motto: Gaussian processes are difficult, Brownian motion is easy.

We show that every separable Gaussian process with integrable variance function admits a Fredholm representation with respect to a Brownian motion.

We extend the Fredholm representation to a transfer principle and develop stochastic analysis by using it. In particular, we prove an Itô formula that is, as far as we know, the most general Skorohod-type Itô formula for Gaussian processes so far.

Finally, we give applications to equivalence in law and series expansions of Gaussian processes.
1 Fredholm Representation

2 Transfer Principle

3 Applications
Outline

1. Fredholm Representation

2. Transfer Principle

3. Applications
**Fredholm Representation**

**Theorem (Fredholm Representation)**

Let \( X = (X_t)_{t \in [0, T]} \) be a separable centered Gaussian process. Then there exists a kernel \( K_T \in L^2([0, T]^2) \) and a Brownian motion \( W = (W_t)_{t \geq 0} \), independent of \( T \), such that

\[
X_t = \int_0^T K_T(t, s) \, dW_s
\]

if and only if the covariance \( R \) of \( X \) satisfies the trace condition

\[
\int_0^T R(t, t) \, dt < \infty.
\]
Fredholm Representation
Some General Remarks

- The Fredholm Kernel $K_T$ usually depends on $T$ even if $R$ does not.
- $K_T$ may be assumed to be symmetric.
- $K_T$ is unique in the sense that if there is another representation with kernel $\tilde{K}_T$, then $\tilde{K}_T = UK_T$ for some unitary operator $U$ on $L^2([0, T])$.
- The Fredholm Representation Theorem holds also for the parameter space $\mathbb{R}_+$, but the trace condition seldom holds, i.e. typically
  \[ \int_0^\infty R(t, t) \, dt = \infty. \]
- If the covariance $R$ is degenerate, one needs to extend the probability space to carry the Brownian motion.
**Fredholm Representation**

**Some Square-Root Remarks**

- $K_T$ (operator) can be constructed from $R_T$ (operator) as the unique positive symmetric square-root, i.e. the operator $K_T$ is a limit of polynomials:

\[
K_T = \lim_{n \to \infty} P_n(R_T).
\]

- The positive symmetric square-root is different from the Cholesky square-root. Indeed, the Cholesky square-root would correspond the Volterra representation

\[
X_t = \int_0^t K(t, s) \, dW_s.
\]

The Volterra representation does not hold for Gaussian processes in general.
Fredholm Representation
Example I

Consider a truncated series expansion

$$X_t = \sum_{k=1}^{n} e_k^T(t)\xi_k,$$

where \(\xi_k\) are independent standard normal random variables and 
\(e_k^T(t) = \int_{0}^{t} \tilde{e}_k^T(s) \, ds\), where \(\tilde{e}_k^T\), \(k \in \mathbb{N}\), is an orthonormal basis in 
\(L^2([0, T])\). This process is not purely non-deterministic and consequently, \(X\) cannot have Volterra representation while \(X\) admits a Fredholm representation. On the other hand, by choosing 
\(\tilde{e}_k^T\) to be the trigonometric basis on \(L^2([0, T])\), \(X\) is a finite-rank approximation of the Karhunen–Loève representation of standard Brownian motion on \([0, T]\). Hence by letting \(n \rightarrow \infty\) we obtain a standard Brownian motion, and hence a Volterra process.
Fredholm Representation
Example II

Let $W$ be a standard Brownian motion and consider the Brownian bridge $B$. The orthogonal representation of $B$ is

$$B_t = W_t - \frac{t}{T} W_T.$$  

Consequently, $B$ has a Fredholm representation with kernel

$$K_T(t, s) = 1_{[0,t]}(s) - \frac{t}{T}.$$  

The canonical representation of the Brownian bridge is

$$B_t = (T - t) \int_0^t \frac{1}{T - s} \, dW_s.$$  

Hence $B$ has also a Volterra representation with kernel

$$K(t, s) = \frac{T - t}{T - s}.$$
Fredholm Representation

The Proof

By the Mercer’s theorem

\[ R(t, s) = \sum_{i=1}^{\infty} \lambda_i^T e_i^T(t)e_i^T(s), \]

where \((\lambda_i^T)_{i=1}^{\infty}\) and \((e_i^T)_{i=1}^{\infty}\) are the eigenvalues and the eigenfunctions of the covariance operator

\[ R_T f(t) = \int_0^T f(s)R(t, s)\,ds. \]

Moreover, \((e_i^T)_{i=1}^{\infty}\) is an orthonormal system on \(L^2([0, T])\).

Since \(R_T\) is a covariance-operator, it admits a square-root operator \(K_T\). By the trace condition \(R_T\) is trace-class, and hence \(K_T\) is Hilbert-Schmidt. Thus, \(K_T\) admits a Kernel.
Fredholm Representation
The Proof

Indeed,

\[ K_T(t, s) = \sum_{i=1}^{\infty} \sqrt{\lambda_i} e_i^T(t)e_i^T(s). \]

Now \( K_T \) is obviously symmetric and we have

\[ R(t, s) = \int_0^T K_T(t, u)K_T(s, u) \, du \]

from which the Fredholm Representation follows by enlarging the probability space, if needed.
Outline

1 Fredholm Representation

2 Transfer Principle

3 Applications
The adjoint operator $\Gamma^*$ of a kernel $\Gamma \in L^2([0, T]^2)$ is defined by linearly extending the relation

$$\Gamma^* 1_{[0, t)} = \Gamma(t, \cdot).$$

**Remark**

If $\Gamma(\cdot, s)$ is of bounded variation for all $s$ and $f$ is nice enough, then

$$\Gamma^* f(s) = \int_0^T f(t) \Gamma(dt, s).$$
Transfer Principle
for Malliavin Derivatives and Skorohod Integrals

Theorem (Transfer Principle)

Let $X$ be Gaussian Fredholm process with kernel $K_T$. Let $D_T$, $\delta_T$, $D^W_T$ and $\delta^W_T$ be the Malliavin derivative and the Skorohod integral with respect to $X$ and to the Brownian motion $W$. Then

$$\delta_T = \delta^W_T K_T^* \quad \text{and} \quad K_T^* D_T = D^W_T.$$ 

Proof: Trivial.
**Theorem (Itô Formula)**

Let $X$ be centered Gaussian process with covariance $R$ and let $f \in C^2$. Then

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) \, \delta X_s + \frac{1}{2} \int_0^t f''(X_s) \, dR(s, s),$$

if anything.
Outline

1. Fredholm Representation

2. Transfer Principle

3. Applications
Let us show how to use the Fredholm Representation and the Transfer Principle to analyze equivalence of Gaussian laws.

Recall the Hitsuda Representation Theorem: A centered Gaussian process $\tilde{W}$ is equivalent to a Brownian motion $W$ if and only if there exists a Volterra kernel $\ell \in L^2([0, T]^2)$ such that

$$d\tilde{W}_t = dW_t + \int_0^t \ell(t, s) dW_s \cdot dt.$$ 

Now, let $\tilde{X}$ and $X$ be Gaussian Fredholm processes with

$$\tilde{X}_t = \int_0^T \tilde{K}_T(t, s) dW_s,$$

$$X_t = \int_0^T K_T(t, s) dW_s.$$
Suppose then that \( \tilde{X} \) has (also) representation

\[
\tilde{X}_t = \int_0^T K_T(t, s) \, d\tilde{W}_s
\]

where \( \tilde{W} \) and \( W \) are equivalent.

Then, obviously \( \tilde{X} \) and \( X \) are equivalent. By plugging in the Hitsuda connection we obtain

\[
\tilde{X}_t = \int_0^T \left[ K_T(t, s) + \int_s^T K_T(t, u)\ell(u, s) \, du \right] \, dW_s.
\]

Thus, we have shown the following:
Applications
Equivalence of Laws

Theorem (Equivalence of Laws)

Let $X$ and $\tilde{X}$ be two Gaussian process with Fredholm kernels $K_T$ and $\tilde{K}_T$, respectively. If there exists a Volterra kernel $\ell \in L^2([0, T]^2)$ such that

$$\tilde{K}_T(t, s) = K_T(t, s) + \int_s^T K_T(t, u)\ell(u, s) \, du,$$

then $X$ and $\tilde{X}$ are equivalent in law.

If the kernel $K_T$ satisfies appropriate non-degeneracy property, then the condition above is also necessary.
Applications
Equivalence of Laws

In the same way, as in the case of equivalence of laws, we see that:

**Theorem (Series representation)**

Let $X$ be a Gaussian Fredholm process with kernel $K_T$ and let $\varphi_k^T$, $k \in \mathbb{N}$, be any orthonormal basis in $L^2([0, T])$. Then

$$X_t = \sum_{k=1}^{\infty} \int_0^T K_T(t, s)\varphi_k^T(s) \, ds \cdot \xi_k,$$

where $\xi_k$, $k \in \mathbb{N}$, are i.i.d. standard Gaussian random variables.

The series above converges in $L^2(\Omega)$; and also almost surely uniformly if and only if $X$ is continuous.
Thank you for listening!
Any questions?