A New characterization of Brownian motion as isotropic i.i.d.-component Lévy process

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Reference

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Abstract

The Brownian motion is arguably the most important stochastic process there is. It has a long history in particle physics dating back to at least to the Roman poet and philosopher Lucretius and his scientific poem De rerum natura ca. 50 BC. Since then, the Brownian motion has proven to be central also in such various fields as physics, economics, quantitative finance, and evolutionary biology, just to mention few.

Abstract

The Brownian motion has many characterizations. It is, for example,

- the scaling limit of random walks,
- the independent-increment stationary-increment Gaussian process,
- the 1/2-self-similar stationary-increment Gaussian process,
- the Markov process with Laplacian as its generator,
- the continuous Lévy process,
- the continuous local martingale with identity function as its bracket.

We provide a new characterization of the Brownian motion as the isotropic Lévy process with i.i.d. components. Our proof will be short and simple, and highly non-elementary.

Outline

1 Brownian motion as Gaussian Process

2 Brownian motion as Lévy Process

3 New Characterization

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Brownian motion as Gaussian Process

Of all the characterizations of the Brownian motion, let us just give here the shortest one:

Definition (BMGP)

A centered stochastic process $W = (W^1, \ldots, W^d)$ with $W_0 = 0$ is the (standard) Brownian motion if it is Gaussian with variance-covariance matrix given by $\mathrm{E}[W_t^i W_s^j] = (t \wedge s)\delta_{ij}$, where $t \wedge s = \min(t, s)$ and

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Brownian motion as Gaussian Process

Definition (BMGP) is of course concise and opaque. We will have better definitions later. Nevertheless, let us show that Definition (BMGP) is not vacuous, i.e., such a process does indeed exist.

Theorem (BMSE)

Let ξ_k^j, $j = 1, \ldots, d$, $k \in \mathbb{N}$, be i.i.d. standard Gaussian random variables. For every $t \in [0, T]$ and $i = 1, \ldots, d$, set

$$W_t^j = \sum_{k=1}^{\infty} \int_0^t e_k(s) \, \mathrm{d}s \; \xi_k^j,$$

where $(e_k)_{k \geq 0}$ is your favorite orthonormal basis on $L^2 = L^2([0, T])$. Then the series above converges in L^2 and the process $W = (W^1, \ldots, W^d)$ is the Brownian motion on the time interval $[0, T]$.

Brownian motion as Gaussian Process

To see that the series above defines the Brownian motion (in the sense of Definition (BMGP)), one simply calculates the covariance. Indeed, let $\langle \cdot, \cdot \rangle$ be the inner product of L^2. Then

$$\begin{aligned}
\mathsf{E}\left[W_t^i W_s^j\right] &= \sum_{k,l=1}^{\infty} \int_0^t e_k(u)\mathrm{d}u \int_0^s e_l(v)\,\mathrm{d}v\,\mathsf{E}[\xi_k^i \xi_l^j] \\
&= \sum_{k=1}^{\infty} \langle 1_{[0,t]}, e_k\rangle \langle 1_{[0,s]}, e_k\rangle \delta_{ij} \\
&= \langle 1_{[0,t]}, 1_{[0,s]}\rangle \delta_{ij} \\
&= (t \wedge s)\delta_{ij}.
\end{aligned}$$

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Brownian motion as Lévy Process

The following is a common TEXTBOOK DEFINITION of the Brownian motion.

Definition (BMTB)

A centered stochastic process $W = (W^1, \dots, W^d)$ with $W_0 = 0$ is the BROWNIAN MOTION if:

1. $W_{t_n} - W_{t_{n-1}}$, $W_{t_{n-1}} - W_{t_{n-2}}$, …, $W_{t_2} - W_{t_1}$ are independent; i.e., the process W has INDEPENDENT INCREMENTS,
2. The distribution of $W_{t+h} - W_t$ does not depend on t; i.e., the process W has STATIONARY INCREMENTS,
3. the process $(W_t)_{t \geq 0}$ has almost surely CONTINUOUS PATHS,
4. $W_{t+h} - W_t$ is centered NORMAL with variance-covariance matrix $h\mathrm{Id}$, where Id is the $d \times d$ identity matrix.

Brownian motion as Lévy Process

Remark (Standard Brownian Motion)

The property (iv) of Definition (BMTB) states that, in addition of being Gaussian, the process W also has independent identically distributed components W^i, $i = 1, \ldots, d$, and $\mathrm{E}[(W_1^i)^2] = 1$. Sometimes one only insists that the process W is Gaussian. Under the assumptions (i)–(iii) of Definition (BMTB) this would mean that $W_{t+h} - W_t$ is a centered Gaussian vector with variance-covariance matrix $h\Sigma$, where Σ is the variance-covariance matrix of W_1. With this more relaxed definition, one usually says that if $\Sigma = \mathrm{Id}$, then W is the standard Brownian motion. The connection between the standard Brownian motion and the relaxed definition of the Brownian motion is simple. Indeed, if W is the standard Brownian motion, and we decompose $\Sigma = \mathrm{K}\mathrm{K}^\top$, then $\mathrm{K}W$ is a Brownian motion in the relaxed sense.

Brownian motion as Lévy Process

Definition (Lévy process)

A stochastic process $L = (L^1, \dots, L^d)$ with $L_0 = 0$ is a Lévy process, if

1. it has independent increments,
2. it has stationary increments,
3. it is stochastically continuous, i.e.,

$$\lim_{h \to 0} \mathrm{P}\left[|L_{t+h} - L_t| > \varepsilon\right] = 0.$$

Brownian motion as Lévy Process

Theorem (Lévy–Khintchine Representation)

A stochastic process $L = (L^1, \dots, L^d)$ is a Lévy process if and only if its characteristic function is of the form

$$\mathsf{E}\left[\mathrm{e}^{\mathrm{i}\langle\theta, L_t\rangle}\right] = \mathrm{e}^{-t\Psi(\theta)},$$

where the characteristic exponent *is of the form*

$$\begin{aligned}\Psi(\theta) &= \mathrm{i}\langle m, \theta\rangle + \frac{1}{2}\langle\theta, \Sigma\theta\rangle \\ &\quad + \int_{\mathbb{R}^d}\left[1 - \mathrm{e}^{\mathrm{i}\langle\theta, x\rangle} + \mathrm{i}\langle\theta, x\rangle 1_{\mathbb{B}^d}(x)\right]\nu(\mathrm{d}x).\end{aligned}$$

Here $m \in \mathbb{R}^d$ is the drift parameter, the symmetric non-negative definite $\Sigma \in \mathbb{R}^{d\times d}$ is the diffusion parameter, and ν, the so-called Lévy measure, *is a measure on $\mathbb{R}^d \setminus \{0\}$.*

Brownian motion as Lévy Process

A Lévy process is continuous if and only if the Lévy measure $\nu \equiv 0$. Consequently, it follows from the Lévy–Khintchine representation that:

Theorem (BMLP)

A centered process W is Brownian motion if and only if it is continuous Lévy process.

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New Characterization

Theorem

Let $d \geq 2$. A Lévy process $W = (W^1, \ldots, W^d)$ is (a multiple of) the standard Brownian motion if and only if it is centered and isotropic with i.i.d. components.

Proof.

The only if part is clear. For the if part, let (m, Σ, ν) be the Lévy triplet of W. Since W has independent components, ν is concentrated on the coordinate axes. Since W is isotropic, ν is also isotropic. Consequently, $\nu \equiv 0$. Since W is centered, $m \equiv 0$. Finally, since W has i.i.d. components, $\Sigma = \sigma \mathrm{Id}$. Thus $(m, \Sigma, \nu) = (0, \sigma \mathrm{Id}, 0)$ proving the claim. □

Thank you for listening!

Any questions?