Representing Gaussian Processes with Martingales

with Application to MLE of Langevin Equation

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Outline

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Gaussian Processes and Martingales

A centered Gaussian process X on $[0, T]$ is completely defined by its covariance function

$$R_X(t,s) \quad = \quad \mathbf{E}\left[X_t X_s\right].$$

This seems simple enough. But for Gaussian martingales M on $[0, T]$ the situation is even simpler. Indeed,

$$R_M(t,s) \quad = \quad \mathbf{E}\left[M_t M_s\right] \quad = \quad \langle M \rangle_{t \wedge s}$$

is a function of one variable only.

Moreover, for Gaussian martingales we have the powerful stochastic calculi of Ito and Malliavin at our disposal.

Gaussian Processes and Martingales

So it seems desirable to connect Gaussian processes with Gaussian martingales to invoke the tools of Ito–Malliavin calculus.

This connection is called the Transfer Principle.

As an example we show how to construct an MLE for the drift parameter of the Langevin equation with relatively general Gaussian noise.

Outline

Invertible Gaussian Volterra Processes and Transfer Principle

A Gaussian process X on $[0, T]$ is Invertible Gaussian Volterra process (IGV) if the exist a Gaussian martingale M with bracket $\langle M \rangle_t = \mathbf{E}[M_t^2]$ such that

$$\begin{aligned} X_t &= \int_0^t k(t,s)\,\mathrm{d}M_s, \\ M_t &= \int_0^t k^{-1}(t,s)\,\mathrm{d}X_s. \end{aligned}$$

For convenience we assume that both X and M are continuous.

The Wiener integral $\int_0^t k^{-1}(t,s)\mathrm{d}X_s$ is defined via a Transfer Principle explained later.

Invertible Gaussian Volterra Processes and Transfer Principle

Remark: A good guess for M is the Prediction Martingale

$$M_t \;=\; \mathbf{E}\left[X_T | \mathcal{F}_t^X\right].$$

Alas, now M depend on T. This is not too bad, or at least unavoidable for the transfer principle in general.

Define the "adjoint" operators K_T^* and $(K_T^*)^{-1}$ by linearly extending the relations

$$\begin{aligned} K_T^* \mathbf{1}_{[0,t]} &= k(t,s), \\ (K_T^*)^{-1} \mathbf{1}_{[0,t]} &= k^{-1}(t,s). \end{aligned}$$

Invertible Gaussian Volterra Processes and Transfer Principle

Then we have the Transfer Principle (TP) for Wiener and Skorohod integrals

$$\begin{aligned}\int_0^T f(t)\,\mathrm{d}X_t &= \int_0^T K_T^* f(t)\,\mathrm{d}M_t,\\ \int_0^T g(t)\,\mathrm{d}M_t &= \int_0^T (K_T^*)^{-1} g(t)\,\mathrm{d}X_t.\end{aligned}$$

Remark: If k is smooth enough and vanishes on the diagonal, then

$$K_T^* f(t) = \int_t^T f(s)\,k(\mathrm{d}s, t).$$

Ditto for $(K^*)^{-1}$.

Outline

MLE for Langevin Equation with Gaussian Noise

We show how to apply the TRANSFER PRINCIPLE for MLE's.

Consider the LANGEVIN EQUATION on $t \in [0, T]$:

$$\mathrm{d}G_t^\theta = -\theta G_t^\theta \,\mathrm{d}t + \mathrm{d}G_t,$$

where θ is the parameter to be estimated and G is a CONTINUOUS process with mean zero.

The solution is

$$G_t^\theta = \mathrm{e}^{-\theta t} \left(G_0 + \int_0^t \mathrm{e}^{\theta s} \,\mathrm{d}G_s \right).$$

The integral can be defined via integration by parts as

$$\int_0^t \mathrm{e}^{\theta s} \,\mathrm{d}G_s = \mathrm{e}^{\theta t} G_t - \int_0^t G_s \,\mathrm{d}\mathrm{e}^{\theta s} = \mathrm{e}^{\theta t} G_t - \theta \int_0^t G_s \mathrm{e}^{\theta s} \,\mathrm{d}s.$$

We have unique pathwise solution if G is continuous. NO ADDITIONAL ASSUMPTIONS ARE NEEDED.

MLE for Langevin Equation with Gaussian Noise

Now assume that G in the Langevin equation is IGV with representations

$$
\begin{aligned}
G_t &= \int_0^t k(t,s)\,\mathrm{d}M_s, \\
M_t &= \int_0^t k^{-1}(t,s)\,\mathrm{d}G_s.
\end{aligned}
$$

Here M is a Gaussian martingale with bracket $\langle M \rangle$ and $k \in L^2([0,T]^2, \mathrm{d}\langle M \rangle)$.

The Wiener integral $\int_0^t k^{-1}(t,s)\mathrm{d}G_s$ is defined via the transfer principle and the kernel $k^{-1}(t,s)$ is assumed to be a true function (as opposed to a generalized function).

MLE for Langevin Equation with Gaussian Noise

Let us then consider MLE for θ.

It makes sense to consider the case $G_0 = G_0^\theta = 0$.

So, we have the equation

$$\mathrm{d}G_t^\theta \quad = \quad -\theta G_t^\theta \,\mathrm{d}t + \mathrm{d}G_t$$

with the solution

$$\begin{aligned} G_t^\theta &= \mathrm{e}^{-\theta t} \int_0^t \mathrm{e}^{\theta s} \,\mathrm{d}G_s \\ &= \mathrm{e}^{\theta t} G_t - \theta \int_0^t G_s \mathrm{e}^{\theta s} \,\mathrm{d}s. \end{aligned}$$

MLE for Langevin Equation with Gaussian Noise

To make MLE non-trivial we must assume the G^θ and G are equivalent in law. Indeed, in the Gaussian case the only alternative would be singularity.

Girsanov–Hitsuda theorem suggest that the bracket $\mathrm{d}\langle M \rangle$ is consistent with the drift $\mathrm{d}t$. Indeed, we have the following NECESSARY AND SUFFICIENT condition for the existence of non-trivial MLE:

Assume that

$$\text{(A)} \qquad Q_t = \frac{\mathrm{d}}{\mathrm{d}\langle M \rangle_t} \int_0^t k^{-1}(t,s) G_s^\theta \, \mathrm{d}s$$

exists in $L^2(\Omega \times [0, T])$.

MLE for Langevin Equation with Gaussian Noise

Let us find the MLE by using a kind of inverse transfer principle.

Start with the Langevin equation

$$\mathrm{d}G_s^\theta = -\theta G_s^\theta \,\mathrm{d}s + \mathrm{d}G_s$$

and integrate against the kernel $k^{-1}(t,s)$ on both sides. This yields

$$\int_0^t k^{-1}(t,s)\,\mathrm{d}G_s^\theta = -\theta \int_0^t k^{-1}(t,s) G_s^\theta \,\mathrm{d}s + \int_0^t k^{-1}(t,s)\,\mathrm{d}G_s$$

Or, by denoting $M_t^\theta = \int_0^t k^{-1}(t,s)\mathrm{d}G_s^\theta$,

$$M_t^\theta = -\theta \int_0^t k^{-1}(t,s) G_s^\theta \,\mathrm{d}s + M_t.$$

MLE for Langevin Equation with Gaussian Noise

Now we use the assumption (A) and obtain

$$\mathrm{d}M_t^\theta = -\theta Q_t \,\mathrm{d}\langle M\rangle_t + \mathrm{d}M_t.$$

So, we have written the linear-drift Langevin equation with noise G as a Langevin equation with non-linear drift and noise M.

By the Girsanov theorem for martingales we can write the log-likelihood as

$$\begin{aligned}\ell_T(\theta) &= \log \frac{\mathrm{d}\mathbf{P}_T^\theta}{\mathrm{d}\mathbf{P}_T} \\ &= -\theta \int_0^T Q_t \,\mathrm{d}M_t^\theta - \frac{\theta^2}{2}\int_0^T Q_t^2 \,\mathrm{d}\langle M\rangle_t.\end{aligned}$$

MLE for Langevin Equation with Gaussian Noise

Maximizing $\ell_T(\theta)$ with respect to θ is trivial.

Indeed,

$$\ell'_T(\theta) \quad = \quad -\int_0^T Q_t \,\mathrm{d}M_t^\theta - \theta \int_0^T Q_t^2 \,\mathrm{d}\langle M \rangle_t.$$

Setting $\ell'_T(\theta) = 0$ yields immediately the MLE

$$\hat{\theta}_T \quad = \quad -\frac{\int_0^T Q_t \,\mathrm{d}M_t^\theta}{\int_0^T Q_t^2 \,\mathrm{d}\langle M \rangle_t}.$$

MLE for Langevin Equation with Gaussian Noise

Let us then consider the Strong Consistency of the MLE. Now

$$
\begin{aligned}
\hat{\theta}_T &= -\frac{\int_0^T Q_t \, \mathrm{d}M_t^{\theta}}{\int_0^T Q_t^2 \, \mathrm{d}\langle M \rangle_t} \\
&= \frac{\int_0^T Q_t \, \{\theta Q_t \, \mathrm{d}\langle M \rangle_t - \mathrm{d}M_t\}}{\int_0^T Q_t^2 \, \mathrm{d}\langle M \rangle_t} \\
&= \frac{\theta \int_0^T Q_t^2 \, \mathrm{d}\langle M \rangle_t - \int_0^T Q_t \, \mathrm{d}M_t}{\int_0^T Q_t^2 \, \mathrm{d}\langle M \rangle_t}
\end{aligned}
$$

Thus,

$$
\hat{\theta}_T - \theta = -\frac{\int_0^T Q_t \, \mathrm{d}M_t}{\int_0^T Q_t^2 \, \mathrm{d}\langle M \rangle_t}.
$$

MLE for Langevin Equation with Gaussian Noise

Note that the downstairs of the difference is nothing but the quadratic variation of the martingale appearing in the upstairs of the difference.

Thus, by the strong law of large numbers for martingales, we obtain that $\hat{\theta}_T$ is strongly consistent if

$$\text{(B)} \qquad \lim_{T\to\infty} \int_0^T Q_t^2 \, \mathrm{d}\langle M\rangle_t = +\infty \quad \text{a.s.}$$

Assumption (B), and assumption (A), and also the IGV assumption, can be written in terms of the covariance of G, but it is complicated.

Outline

Invertible Gaussian Fredholm Processes and Transfer Principle

It follows from the Mercer's theorem that any centered Gaussian process X with $X_0 = 0$ on $[0, T]$ with continuous (say) covariance function admits the Fredholm representation

$$X_t = \int_0^T k_T(t,s)\,\mathrm{d}W_s,$$

where W is a Brownian motion.

Obviously, in general there is no hope in having the inversion

$$W_t = \int_0^T k_T^{-1}(t,s)\,\mathrm{d}X_s.$$

Indeed, X can be, e.g., generated by a single Gaussian variable.

Invertible Gaussian Fredholm Processes and Transfer Principle

There seems to be hope, when one replaces the Brownian motion W with a Gaussian martingale M that generates dynamically the same amount of randomness as X, whatever that means. So, with "MINIMAL ASSUMPTIONS" one hopes to get

$$\begin{aligned} X_t &= \int_0^T k_T(t,s)\,\mathrm{d}M_s, \\ M_t &= \int_0^T k_T^{-1}(t,s)\,\mathrm{d}X_s. \end{aligned}$$

The obvious problem here is that the KERNELS k_T AND k_T^{-1} DEPEND ON T. This is, I think, a minor inconvenience. Indeed, the"adjoint" operators K_T^* and $(K_T^*)^{-1}$ depend on T even in the Volterra case.

The TRANSFER PRINCIPLE works pretty much the same way in the Fredholm case as in the Volterra case.

Thank you for listening!

Any questions?