Local Continuity

Possibly an Old and Uninteresting Concept Per Se

Tommi Sottinen

Reykjavík University

25th October 2007

Outline

Outline

Definition (Local Continuity)

Let $\mathcal{X}$ and $\mathcal{Y}$ be metric spaces. A function $f : \mathcal{X} \to \mathcal{Y}$ is locally continuous if for all $x \in \mathcal{X}$ there exists an open $U_x \subset \mathcal{X}$ such that $x \in \bar{U}_x$ and $f(x_n) \to f(x)$ whenever $x_n \to x$ in U_x.

Definition (Local Continuity)

Let $\mathcal{X}$ and $\mathcal{Y}$ be metric spaces. A function $f : \mathcal{X} \to \mathcal{Y}$ is locally continuous if for all $x \in \mathcal{X}$ there exists an open $U_x \subset \mathcal{X}$ such that $x \in \bar{U}_x$ and $f(x_n) \to f(x)$ whenever $x_n \to x$ in U_x.

Remark (Local, Directional, and Proper Continuity)

Local continuity at x is continuity from the direction U_x. If $x \in U_x$ then local continuity is continuity.

Definition (Local Continuity)

Let $\mathcal{X}$ and $\mathcal{Y}$ be metric spaces. A function $f : \mathcal{X} \to \mathcal{Y}$ is LOCALLY CONTINUOUS if for all $x \in \mathcal{X}$ there exists an open $U_x \subset \mathcal{X}$ such that $x \in \bar{U}_x$ and $f(x_n) \to f(x)$ whenever $x_n \to x$ in U_x.

Remark (Local, Directional, and Proper Continuity)

Local continuity at x is continuity from the direction U_x. If $x \in U_x$ then local continuity is continuity.

Remark (Generalization to Topological Spaces)

One might want to generalize the concept of Local Continuity to topological (measure) spaces.

EXAMPLE (SIMPLE ONE)

An indicator $\mathbf{1}_A : \mathbb{R} \to \mathbb{R}$

1. is locally continuous if $A = \bar{G}$, G is open,
2. is not locally continuous if A has an isolated point.

EXAMPLE (SIMPLE ONE)

An indicator $\mathbf{1}_A : \mathbb{R} \to \mathbb{R}$

1. is locally continuous if $A = \bar{G}$, G is open,
2. is not locally continuous if A has an isolated point.

EXAMPLE (INTERESTING ONE)

A functional $\tau : C[0, T] \to [0, T]$ defined by

$$\tau(\omega) = \min\{t; \omega(t) = c\}$$

is locally continuous.

Example (Simple One)

An indicator $\mathbf{1}_A : \mathbb{R} \to \mathbb{R}$

1. is locally continuous if $A = \bar{G}$, G is open,
2. is not locally continuous if A has an isolated point.

Example (Interesting One)

A functional $\tau : C[0, T] \to [0, T]$ defined by

$$\tau(\omega) = \min \{t; \omega(t) = c\}$$

is locally continuous. Indeed, for $\omega_0 \in C[0, T]$, take

$$U_{\omega_0} = \{\omega; \omega(t) > \omega_0(t) \text{ for all } t \in [0, T]\} .$$

Example

Consider functions $f : \mathbb{R}^2 \to \mathbb{R}$.

Example

Consider functions $f : \mathbb{R}^2 \to \mathbb{R}$.

1.

$$f(x, y) = \mathbf{1}_{\{0\}\times[0,\infty)}(x, y)$$

is directionally continuous at $(0, 0)$ along path $\{(0, y); y \geq 0\}$, but not locally continuous at $(0, 0)$.

Example

Consider functions $f : \mathbb{R}^2 \to \mathbb{R}$.

1. $$f(x, y) = \mathbf{1}_{\{0\}\times[0,\infty)}(x, y)$$

   is directionally continuous at $(0, 0)$ along path $\{(0, y); y \geq 0\}$, but not locally continuous at $(0, 0)$.

2. $$f(x, y) = \sum_{n=1}^{\infty} \mathbf{1}_{\left(-1,1\right)\times\left(2^{-4(n+1)},2^{-4n}\right)}(x, y)$$

   is locally continuous at $(0, 0)$ but not directionally continuous along any path ending at $(0, 0)$.

Outline

Definition (Stopping Time)

Let $(\mathcal{F}_t)_{t\in[0,T]}$ be a flow of information. A random variable $\tau : \Omega \to [0, T]$ is an $(\mathcal{F}_t)$-stopping time if $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \in [0, T]$.

Definition (Stopping Time)

Let $(\mathcal{F}_t)_{t \in [0,T]}$ be a flow of information. A random variable $\tau : \Omega \to [0, T]$ is an $(\mathcal{F}_t)$-stopping time if $\{\tau \le t\} \in \mathcal{F}_t$ for all $t \in [0, T]$.

Example

Let $(\mathcal{F}_t)$ be the information generated by observing a stochastic process (S_t). Then

Definition (Stopping Time)

Let $(\mathcal{F}_t)_{t\in[0,T]}$ be a flow of information. A random variable $\tau : \Omega \to [0, T]$ is an $(\mathcal{F}_t)$-stopping time if $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \in [0, T]$.

Example

Let $(\mathcal{F}_t)$ be the information generated by observing a stochastic process (S_t). Then

1. $\tau(\omega) = \inf\{t; S_t(\omega) \geq c\}$ is a stopping time,

Definition (Stopping Time)

Let $(\mathcal{F}_t)_{t\in[0,T]}$ be a flow of information. A random variable $\tau : \Omega \to [0, T]$ is an $(\mathcal{F}_t)$-stopping time if $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \in [0, T]$.

Example

Let $(\mathcal{F}_t)$ be the information generated by observing a stochastic process (S_t). Then

1. $\tau(\omega) = \inf\{t; S_t(\omega) \geq c\}$ is a stopping time,
2. $\tau(\omega) = \inf\{t; S_t(\omega) = \max_{u\in[0,T]} S_u(\omega)\}$ is not a stopping time.

The following stopping times $\tau : C[0, T] \rightarrow [0, T]$ are locally continuous.

The following stopping times $\tau : C[0, T] \to [0, T]$ are locally continuous.

Example

The following stopping times $\tau : C[0, T] \to [0, T]$ are locally continuous.

Example

1. $\tau(\omega) = \inf\{t; \omega(t) \in F\}$, F is closed,

The following stopping times $\tau : C[0, T] \to [0, T]$ are locally continuous.

Example

1. $\tau(\omega) = \inf\{t; \omega(t) \in F\}$, F is closed,
2. $\tau(\omega) = \inf\{t; \psi(t, \omega) \in \bar{G}\}$, ψ is continuous and G is open,

The following stopping times $\tau : C[0, T] \to [0, T]$ are locally continuous.

Example

1. $\tau(\omega) = \inf\{t; \omega(t) \in F\}$, F is closed,
2. $\tau(\omega) = \inf\{t; \psi(t, \omega) \in \bar{G}\}$, ψ is continuous and G is open,
3. $\tau(\omega) = \inf\{t; (t, \omega) \in \bar{\mathcal{U}}\}$, $\mathcal{U}$ is open.

The following stopping times $\tau : C[0, T] \to [0, T]$ are locally continuous.

Example

1. $\tau(\omega) = \inf\{t; \omega(t) \in F\}$, F is closed,
2. $\tau(\omega) = \inf\{t; \psi(t, \omega) \in \bar{G}\}$, ψ is continuous and G is open,
3. $\tau(\omega) = \inf\{t; (t, \omega) \in \bar{\mathcal{U}}\}$, $\mathcal{U}$ is open.

The functionals in the example above are locally continuous even if they were not stopping times.

Outline

Let $S = (S_t)_{t \in [0,T]}$ be an asset-price process. We consider the canonical probability space, where $\Omega = C_+[0, T]$, $\mathcal{F}$ is its Borel-σ-algebra, and $\mathbf{P}$ is the distribution of S. So we have $S_t(\omega) = \omega(t)$.

Let $S = (S_t)_{t \in [0,T]}$ be an asset-price process. We consider the canonical probability space, where $\Omega = C_+[0,T]$, $\mathcal{F}$ is its Borel-σ-algebra, and $\mathbf{P}$ is the distribution of S. So we have $S_t(\omega) = \omega(t)$.

Definition (Option)

Option is simply a mapping $G : C_+[0,T] \to \mathbb{R}$. The asset S is the underlying of the option G.

Let $S = (S_t)_{t \in [0,T]}$ be an asset-price process. We consider the canonical probability space, where $\Omega = C_+[0,T]$, $\mathcal{F}$ is its Borel-σ-algebra, and $\mathbf{P}$ is the distribution of S. So we have $S_t(\omega) = \omega(t)$.

Definition (Option)

Option is simply a mapping $G : C_+[0,T] \to \mathbb{R}$. The asset S is the underlying of the option G.

Example

- $G = (S_T - K)^+$ is a call-option,
- $G = (K - S_T)^+$ is a put-option,
- $G = S_T - K$ is a future.

A trading strategy $\Phi = (\Phi_t)_{t \in [0,T]}$ is an S-adapted stochastic process that tells the units of the underlying asset S the investor has is her portfolio at any time $t \in [0, T]$.

A trading strategy $\Phi = (\Phi_t)_{t \in [0,T]}$ is an S-adapted stochastic process that tells the units of the underlying asset S the investor has is her portfolio at any time $t \in [0, T]$.

The wealth of the trading strategy Φ is (in the discounted world) satisfies

$$\mathrm{d}V_t(\Phi) \;=\; \Phi_t \,\mathrm{d}S_t,$$

where the differentials are of "forward type".

A TRADING STRATEGY $\Phi = (\Phi_t)_{t\in[0,T]}$ is an S-adapted stochastic process that tells the units of the underlying asset S the investor has is her portfolio at any time $t \in [0, T]$.

The WEALTH of the trading strategy Φ is (in the discounted world) satisfies

$$\mathrm{d}V_t(\Phi) \;=\; \Phi_t \,\mathrm{d}S_t,$$

where the differentials are of "forward type".

DEFINITION (ARBITRAGE)

ARBITRAGE is a trading strategy Φ with the properties: $V_0(\Phi) = 0,\ V_t(\Phi) \geq 0$ for all $t \in [0, T]$, and $\mathbf{P}[V_T(\Phi) > 0] > 0$.

A TRADING STRATEGY $\Phi = (\Phi_t)_{t\in[0,T]}$ is an S-adapted stochastic process that tells the units of the underlying asset S the investor has is her portfolio at any time $t \in [0, T]$.

The WEALTH of the trading strategy Φ is (in the discounted world) satisfies

$$\mathrm{d}V_t(\Phi) \;=\; \Phi_t \,\mathrm{d}S_t,$$

where the differentials are of "forward type".

Definition (Arbitrage)

ARBITRAGE is a trading strategy Φ with the properties: $V_0(\Phi) = 0$, $V_t(\Phi) \geq 0$ for all $t \in [0, T]$, and $\mathbf{P}[V_T(\Phi) > 0] > 0$.

It is an economic axiom that there should be no arbitrage.

Replication principle is used to hedge and price options.

Replication principle is used to hedge and price options.

Definition (Replication principle)

Let G be an option. Suppose that there is a trading strategy Φ with initial wealth $V_0(\Phi)$ such that $G = V_T(\Phi)$. Then the price of the option G is $V_0(\Phi)$.

Replication principle is used to hedge and price options.

Definition (Replication principle)

Let G be an option. Suppose that there is a trading strategy Φ with initial wealth $V_0(\Phi)$ such that $G = V_T(\Phi)$. Then the price of the option G is $V_0(\Phi)$.

The replication requirement $G = V_T(\Phi)$ can be written as

$$G \;=\; V_0(\Phi) + \int_0^T \Phi_t \,\mathrm{d}S_t,$$

where the integral is of "forward type".

Outline

Assume that S is continuous, strictly positive, starts from s_0, and the information used in trading is generated by it.

Assume that S is continuous, strictly positive, starts from s_0, and the information used in trading is generated by it.

Assume that S has the QUADRATIC VARIATION

$$(\mathrm{d}S_t)^2 \;=\; \sigma^2 S_t^2 \mathrm{d}t.$$

Assume that S is continuous, strictly positive, starts from s_0, and the information used in trading is generated by it.

Assume that S has the QUADRATIC VARIATION

$$(\mathrm{d}S_t)^2 \;=\; \sigma^2 S_t^2 \mathrm{d}t.$$

Assume the CONDITIONAL SMALL-BALL PROPERTY

$$\mathbf{P}\left[\sup_{t\in[\tau,T]} |S_t - \omega(t)| < \varepsilon \,\middle|\, \mathcal{F}_\tau\right] > 0$$

P-a.s. for all paths ω, positive ε, and stopping times τ.

Assume that S is continuous, strictly positive, starts from s_0, and the information used in trading is generated by it.

Assume that S has the QUADRATIC VARIATION

$$(\mathrm{d}S_t)^2 \;=\; \sigma^2 S_t^2 \mathrm{d}t.$$

Assume the CONDITIONAL SMALL-BALL PROPERTY

$$\mathbf{P}\left[\sup_{t\in[\tau,T]} |S_t - \omega(t)| < \varepsilon \,\middle|\, \mathcal{F}_\tau\right] > 0$$

P-a.s. for all paths ω, positive ε, and stopping times τ.

So, we have a collection of models **P** on the canonical filtered space $\mathcal{C}_{s_0,\sigma}[0,T]$, where **P** is restricted only by the assumptions of quadratic variation and conditional small-ball property.

[BSV][1] showed that with ALLOWED strategies that depend smoothly on time, spot, running maximum, running minimum and such one cannot do arbitrage.

[1] Bender, S., Valkeila: *No-arbitrage pricing beyond semimartingales.* WIAS Preprint No. 1110, 2006.

[BSV][1] showed that with ALLOWED strategies that depend smoothly on time, spot, running maximum, running minimum and such one cannot do arbitrage.

The result followed from the fact that

$$V_t(\Phi)(\omega) = V_0(\Phi)(\omega) + v(t,\omega;\varphi) \quad \text{for } \mathbf{P}\text{-a.a. } \omega,$$

where $v(t,\omega;\varphi)$ is continuous in ω uniformly in t. Here φ is the strategy functional associated to Φ:

$$\Phi_t(\omega) = \varphi\Big(t, \omega(t), g_1(t,\omega), \ldots, g_m(t,\omega)\Big),$$

where φ is smooth and $g_1, \ldots, g_m$ are HINDSIGHT FACTORS.

[1] Bender, S., Valkeila: *No-arbitrage pricing beyond semimartingales.* WIAS Preprint No. 1110, 2006.

The allowed strategies are natural from the hedging point of view: Hedging strategies of typical options are of this type. However, from the no-arbitrage point of view the allowed strategies are not so natural: They do not include stopping times.

The allowed strategies are natural from the hedging point of view: Hedging strategies of typical options are of this type. However, from the no-arbitrage point of view the allowed strategies are not so natural: They do not include stopping times.

We can extend the no-arbitrage result of [BSV] to strategies that include certain kind of stopping times. The key concept is local continuity.

The allowed strategies are natural from the hedging point of view: Hedging strategies of typical options are of this type. However, from the no-arbitrage point of view the allowed strategies are not so natural: They do not include stopping times.

We can extend the no-arbitrage result of [BSV] to strategies that include certain kind of stopping times. The key concept is local continuity.

While stopping times are rarely continuous, the author is not aware of any (reasonable) stopping time that is not locally continuous.

Definition (Stopping-Allowed Strategies)

A trading strategy Φ is Stopping-Allowed if it is of the form

$$\Phi_t = \sum_{k=1}^{n} \Phi_t^{(k)} \mathbf{1}_{(\tau_k, \tau_{k+1}]}(t),$$

where the $\Phi^{(k)}$'s are allowed and τ_k's are locally continuous.

Definition (Stopping-Allowed Strategies)

A trading strategy Φ is Stopping-Allowed if it is of the form

$$\Phi_t = \sum_{k=1}^{n} \Phi_t^{(k)} \mathbf{1}_{(\tau_k, \tau_{k+1}]}(t),$$

where the $\Phi^{(k)}$'s are allowed and τ_k's are locally continuous.

The definition above is understood in the conditional sense, i.e. $\Phi^{(k)}$ may depend on on $\mathcal{F}_{\tau_k}$ and $\tau_{k+1} \geq \tau_k$ is locally continuous in the conditioned, or quotient, space $\mathcal{C}_{S_{\tau_k}, \sigma}[\tau_k, T]$.

Theorem (No-Arbitrage with Stopping-Allowed Strategies)

Let Φ be a stopping-allowed strategy. Then Φ is not an arbitrage opportunity.

Theorem (No-Arbitrage with Stopping-Allowed Strategies)

Let Φ be a stopping-allowed strategy. Then Φ is not an arbitrage opportunity.

Theorem (No-Arbitrage with Stopping-Allowed Strategies) follows by applying the conditional small-ball property n times with the following lemma:

Theorem (No-Arbitrage with Stopping-Allowed Strategies)

Let Φ be a stopping-allowed strategy. Then Φ is not an arbitrage opportunity.

Theorem (No-Arbitrage with Stopping-Allowed Strategies) follows by applying the conditional small-ball property n times with the following lemma:

Lemma (No-Arbitrage with Take-the-Money-and-Run Strategies)

Let Φ be allowed strategy and let τ be a locally continuous stopping time. Then $\Phi \mathbf{1}_{[0,\tau]}$ is not an arbitrage opportunity.

Proof of Lemma (No-Arbitrage with Take-the-Money-and-Run Strategies).

Proof of Lemma (No-Arbitrage with Take-the-Money-and-Run Strategies).

Let $\Phi\mathbf{1}_{[0,\tau]}$ be a candidate for an arbitrage opportunity: $V_0(\Phi\mathbf{1}_{[0,\tau]}) = 0$ and $V_T(\Phi\mathbf{1}_{[0,\tau]}) \geq 0$ $\mathbf{P}$-a.s., or

$$v(\tau(\omega), \omega; \varphi) \geq 0 \quad \text{for } \mathbf{P}\text{-a.a. } \omega.$$

Proof of Lemma (No-Arbitrage with Take-the-Money-and-Run Strategies).

Let $\Phi\mathbf{1}_{[0,\tau]}$ be a candidate for an arbitrage opportunity: $V_0(\Phi\mathbf{1}_{[0,\tau]}) = 0$ and $V_T(\Phi\mathbf{1}_{[0,\tau]}) \geq 0$ **P**-a.s., or

$$v(\tau(\omega), \omega; \varphi) \geq 0 \quad \text{for } \mathbf{P}\text{-a.a. } \omega.$$

We show that $v(\tau(\omega), \omega; \varphi) \geq 0$ for all ω:

PROOF OF LEMMA (NO-ARBITRAGE WITH TAKE-THE-MONEY-AND-RUN STRATEGIES).

Let $\Phi\mathbf{1}_{[0,\tau]}$ be a candidate for an arbitrage opportunity: $V_0(\Phi\mathbf{1}_{[0,\tau]}) = 0$ and $V_T(\Phi\mathbf{1}_{[0,\tau]}) \geq 0$ $\mathbf{P}$-a.s., or

$$v(\tau(\omega), \omega; \varphi) \geq 0 \quad \text{for } \mathbf{P}\text{-a.a. } \omega.$$

We show that $v(\tau(\omega), \omega; \varphi) \geq 0$ for all ω: Suppose that $v(\tau(\omega_0), \omega_0; \varphi) < 0$ for some ω_0.

PROOF OF LEMMA (NO-ARBITRAGE WITH TAKE-THE-MONEY-AND-RUN STRATEGIES).

Let $\Phi\mathbf{1}_{[0,\tau]}$ be a candidate for an arbitrage opportunity: $V_0(\Phi\mathbf{1}_{[0,\tau]}) = 0$ and $V_T(\Phi\mathbf{1}_{[0,\tau]}) \geq 0$ $\mathbf{P}$-a.s., or

$$v(\tau(\omega), \omega; \varphi) \geq 0 \quad \text{for } \mathbf{P}\text{-a.a. } \omega.$$

We show that $v(\tau(\omega), \omega; \varphi) \geq 0$ for all ω: Suppose that $v(\tau(\omega_0), \omega_0; \varphi) < 0$ for some ω_0. Let U_{ω_0} be the local continuity set of τ at ω_0. Since $v(t, \cdot; \varphi)$ is continuous uniformly in t we see that $v(\tau(\cdot), \cdot; \varphi)$ is continuous on U_{ω_0}.

Proof of Lemma (No-Arbitrage with Take-the-Money-and-Run Strategies).

Let $\Phi\mathbf{1}_{[0,\tau]}$ be a candidate for an arbitrage opportunity: $V_0(\Phi\mathbf{1}_{[0,\tau]}) = 0$ and $V_T(\Phi\mathbf{1}_{[0,\tau]}) \geq 0$ $\mathbf{P}$-a.s., or

$$v(\tau(\omega), \omega; \varphi) \geq 0 \quad \text{for } \mathbf{P}\text{-a.a. } \omega.$$

We show that $v(\tau(\omega), \omega; \varphi) \geq 0$ for all ω: Suppose that $v(\tau(\omega_0), \omega_0; \varphi) < 0$ for some ω_0. Let U_{ω_0} be the local continuity set of τ at ω_0. Since $v(t, \cdot; \varphi)$ is continuous uniformly in t we see that $v(\tau(\cdot), \cdot; \varphi)$ is continuous on U_{ω_0}. So, there must be a ball $B \subset U_{\omega_0}$ such that $v(\tau(\omega), \omega; \varphi) < 0$ for all $\omega \in B$.

Proof of Lemma (No-Arbitrage with Take-the-Money-and-Run Strategies).

Let $\Phi\mathbf{1}_{[0,\tau]}$ be a candidate for an arbitrage opportunity: $V_0(\Phi\mathbf{1}_{[0,\tau]}) = 0$ and $V_T(\Phi\mathbf{1}_{[0,\tau]}) \geq 0$ $\mathbf{P}$-a.s., or

$$v(\tau(\omega), \omega; \varphi) \geq 0 \quad \text{for } \mathbf{P}\text{-a.a. } \omega.$$

We show that $v(\tau(\omega), \omega; \varphi) \geq 0$ for all ω: Suppose that $v(\tau(\omega_0), \omega_0; \varphi) < 0$ for some ω_0. Let U_{ω_0} be the local continuity set of τ at ω_0. Since $v(t, \cdot; \varphi)$ is continuous uniformly in t we see that $v(\tau(\cdot), \cdot; \varphi)$ is continuous on U_{ω_0}. So, there must be a ball $B \subset U_{\omega_0}$ such that $v(\tau(\omega), \omega; \varphi) < 0$ for all $\omega \in B$. But due to the small-ball property this means that $\mathbf{P}[V_T(\Phi\mathbf{1}_{[0,\tau]}) < 0] > 0$, which is a contradiction.

Proof of Lemma (No-Arbitrage with Take-the-Money-and-Run Strategies), contd.

Since $v(\tau(\omega), \omega; \varphi) \geq 0$ for all ω we have in particular that $V_T(\Phi \mathbf{1}_{[0,\tau]}) \geq 0$ $\tilde{\mathbf{P}}$-a.s. ($\tilde{\mathbf{P}}$ stands for the Black-Scholes reference model).

Proof of Lemma (No-Arbitrage with Take-the-Money-and-Run Strategies), contd.

Since $v(\tau(\omega), \omega; \varphi) \geq 0$ for all ω we have in particular that $V_T(\Phi \mathbf{1}_{[0,\tau]}) \geq 0$ $\tilde{\mathbf{P}}$-a.s. ($\tilde{\mathbf{P}}$ stands for the Black-Scholes reference model). The classical theory then tells us that $V_T(\Phi \mathbf{1}_{[0,\tau]}) = 0$ $\tilde{\mathbf{P}}$-a.s.

Proof of Lemma (No-Arbitrage with Take-the-Money-and-Run Strategies), contd.

Since $v(\tau(\omega), \omega; \varphi) \geq 0$ for all ω we have in particular that $V_T(\Phi \mathbf{1}_{[0,\tau]}) \geq 0$ $\tilde{\mathbf{P}}$-a.s. ($\tilde{\mathbf{P}}$ stands for the Black-Scholes reference model). The classical theory then tells us that $V_T(\Phi \mathbf{1}_{[0,\tau]}) = 0$ $\tilde{\mathbf{P}}$-a.s. Then, by using the local continuity as before, we see that $v(\tau(\omega), \omega; \varphi) = 0$ for all ω.

Proof of Lemma (No-Arbitrage with Take-the-Money-and-Run Strategies), contd.

Since $v(\tau(\omega), \omega; \varphi) \geq 0$ for all ω we have in particular that $V_T(\Phi \mathbf{1}_{[0,\tau]}) \geq 0$ $\tilde{\mathbf{P}}$-a.s. ($\tilde{\mathbf{P}}$ stands for the Black-Scholes reference model). The classical theory then tells us that $V_T(\Phi \mathbf{1}_{[0,\tau]}) = 0$ $\tilde{\mathbf{P}}$-a.s. Then, by using the local continuity as before, we see that $v(\tau(\omega), \omega; \varphi) = 0$ for all ω. But this means that $V(\Phi \mathbf{1}_{[0,\tau]}) = 0$ $\mathbf{P}$-a.s. So, $\Phi \mathbf{1}_{[0,\tau]}$ is not an arbitrage opportunity. $\square$

Proof of Theorem (No-Arbitrage with Stopping-Allowed Strategies).

By using the conditional small-ball property instead of an unconditional one we see that Lemma (No-Arbitrage with Take-the-Money-and-Run Strategies) can be strengthened to:

$$\Phi^{(k)}\mathbf{1}_{(\tau_k,\tau_{k+1}]}$$

is not an arbitrage opportunity. Here the allowed strategy $\Phi^{(k)}$ may depend additionally on $\mathcal{F}_{\tau_k}$, and τ_{k+1} is locally continuous on the quotient, or conditioned, space $\mathcal{C}_{S_{\tau_k},\sigma}[\tau_k, T]$.

Proof of Theorem (No-Arbitrage with Stopping-Allowed Strategies).

By using the conditional small-ball property instead of an unconditional one we see that Lemma (No-Arbitrage with Take-the-Money-and-Run Strategies) can be strengthened to:

$$\Phi^{(k)}\mathbf{1}_{(\tau_k,\tau_{k+1}]}$$

is not an arbitrage opportunity. Here the allowed strategy $\Phi^{(k)}$ may depend additionally on $\mathcal{F}_{\tau_k}$, and τ_{k+1} is locally continuous on the quotient, or conditioned, space $\mathcal{C}_{S_{\tau_k},\sigma}[\tau_k, T]$.

But this means that the stopping-allowed strategy Φ does not generate arbitrage on any of the stochastic intervals $(\tau_k, \tau_{k+1}]$. Hence, it cannot generate arbitrage on the interval $[0, T]$. $\square$

- The End -