# Fractional Brownian Motion as a Model in Finance

Tommi Sottinen, University of Helsinki

Esko Valkeila, University of Turku and University of Helsinki

# Black & Scholes pricing model

In the classical Black & Scholes pricing model the randomness of the stock price $S$ is due to Brownian motion $W$:

$$dS_t = S_t(\mu dt + \sigma dW_t), \quad S_0 > 0.$$

The bond price is $B_t = e^{rt}$.

Parameters $\mu \in \mathbb{R}$, $r, \sigma \in \mathbb{R}_+$ supposed to be known.

Traditionally one assumes that there are no dividends, no transaction costs, same interest rate $r$ for lending and saving on the bond and no limitations on short-selling of the stock.

# B-S model, cont.

Some properties of this pricing model:

- The model is arbitrage free.

- One can give a unique price for options on the stock $S$, e.g. the fair price of an European call-option $(S_T - K)^+$ is

$$S_0 \Phi(y_1) - K e^{-rT} \Phi(y_2), \qquad (1)$$

  where

$$\begin{aligned} y_1 &= \frac{\log \frac{S_0}{K} + rT + \frac{\sigma^2}{2} T}{\sigma T^{1/2}}, \\ y_2 &= \frac{\log \frac{S_0}{K} + rT - \frac{\sigma^2}{2} T}{\sigma T^{1/2}}. \end{aligned}$$

- One can hedge options using the Ito-Clark-Ocone formula.

# Discussion

According to the B & S - model the log-returns

$$R_t := \log \frac{S_t}{S_{t-1}}$$

should be independent normal variables.

- The dependence structure of the log-returns have been studied using the Hurst parameter $H$. In the independent case one should have $H = \frac{1}{2}$. However, some studies show that $H \sim .6$.

- There are empirical studies indicating that the log-returns are not normal.

To overcome with the first critical point, it has been proposed that one should replace the Brownian motion $W$ by *fractional Brownian motion*.

[We will ignore completely the second critical point in what follows.]

# Fractional Brownian motion

Fractional Brownian motion $Z$ is a continuous and centered Gaussian process with stationary increments and variance

$$\mathbb{E}Z_t^2 = t^{2H}.$$

The parameter $H$ allows us to model the statistical long-range dependence of the log-returns. In financial modeling it is assumed that $\frac{1}{2} < H < 1$.

Replace $W$ with $Z$ and consider the following dynamics for the stock price $S$

$$dS_t = S_t(\mu dt + \sigma dZ_t). \tag{2}$$

The solution to (2) is called a geometric fractional Brownian motion.

# Fractional B-S model

Problems:

(a) How to define the stochastic integral (2)?

(b) Is the modified pricing model arbitrage free?

(c) Is the modified pricing model complete, i.e. is there a fractional analogue of the Ito-Clark-Ocone formula?

For the problem (a) two possible definitions: path-wise definition and definition based on generalized stochastic processes.

# fractional B - S model, cont.

Path-wise solution to (2) is

$$S_t = S_0 \exp(\mu t + \sigma Z_t)$$

and generalized solution is

$$S_t = S_0 \exp(\mu t - \frac{\sigma^2}{2} t^{2H} + \sigma Z_t).$$

With generalized solutions to (2) the fractional pricing model is arbitrage free and complete, i.e. problems (b) and (c) are solved [HØ]. However, it seems to be difficult to give an economical interpretation to the formulas.

With path-wise solutions to (2) and with *continuous* trading one can do arbitrage in the fractional pricing model [C,Sh]. Surprisingly, the arbitrage arises in a modified binomial approximation [So].

# European options in fractional models

Using a weak pricing principle one can compute the prices of European options in the path-wise fractional model [V]. These prices coincide with the ones obtained in the generalized model [HØ].

E.g. the price of an call-option $(S_T - K)^+$ is

$$S_0\Phi(y_1(H)) - Ke^{-rT}\Phi(y_2(H)), \qquad (3)$$

where

$$y_1(H) = \frac{\log\frac{S_0}{K} + rT + \frac{\sigma^2}{2}T^{2H}}{\sigma T^H},$$
$$y_2(H) = \frac{\log\frac{S_0}{K} + rT - \frac{\sigma^2}{2}T^{2H}}{\sigma T^H}.$$

Note that (3) converges to (1) as $H \to \frac{1}{2}$.

# Other topics

- Mixed models: $W \to W + Z$ [C,MV].

- Regularizations of the fractional Brownian motion [C].

# Literature

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