Local Continuity of Stopping Times And Arbitrage

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Vienna, September 17-22, 2007

Workshop and Mid-Term Conference on Advanced Mathematical Methods for Finance

Outline

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1. Quadratic Variation Market Models with Conditional Small-Ball Property
2. No-Arbitrage with Allowed Strategies
3. Locally Continuous Stopping Times
4. No-Arbitrage with Stopping-Allowed Strategies
5. No-Arbitrage with Simple Strategies

Quadratic Variation Market Models with Conditional Small-Ball Property

We assume that the stock-price process $S = (S_t)_{t \in [0,T]}$ is (almost surely) continuous, strictly positive, starts from s_0, and the information used in trading is generated by it.

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So, we work in the canonical space $\Omega = \mathcal{C}_{s_0,\sigma}[0,T]$ with $S_t(\eta) = \eta(t)$ and

$$\mathcal{F}_t = \sigma\left\{\eta(s); s \leq t\right\},$$

$\mathcal{F} = \mathcal{F}_T$. (The index $\sigma > 0$ will be explained in the next slide.)

Quadratic Variation Market Models with Conditional Small-Ball Property

We assume that almost surely the stock-price process has the QUADRATIC VARIATION of the Black-Scholes model:

$$\mathrm{d}\langle S\rangle_t = \sigma^2 S_t^2 \mathrm{d}t.$$

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We assume that the following CONDITIONAL SMALL-BALL PROPERTY is satisfied:

$$\mathbf{P}\left[\sup_{t\in[\tau,T]} |S_t - \eta(t)| < \varepsilon \,\middle|\, \mathcal{F}_\tau\right] > 0$$

P-a.s. for all paths η, positive ε, and stopping times τ.

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So, we have a collection of models **P** on the canonical filtered space $\mathcal{C}_{s_0,\sigma}[0,T]$, where **P** is restricted only by the assumptions of quadratic variation and conditional small-ball property.

Outline

No-Arbitrage with Allowed Strategies

In [BSV] (Bender, Sottinen, Valkeila: No-arbitrage pricing beyond semimartingales. WIAS Preprint No. 1110, 2006) we showed that with strategies that depend in a smooth way on time, spot, running maximum, running minimum and such one cannot make arbitrage in quadratic-variation small-ball models. These strategies were called ALLOWED

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The no-arbitrage result followed basically from the fact that we can write the value $V_t(\Phi)(\eta)$ of an allowed strategy (almost surely) by using a value functional $v(t,\eta;\varphi)$:

$$V_t(\Phi)(\eta) = V_0(\Phi)(\eta) + v(t,\eta;\varphi) \quad \text{for } \mathbf{P}\text{-a.a. } \eta,$$

and $v(t,\eta;\varphi)$ is continuous in η uniformly in t. Here φ is the strategy functional associated to Φ:

$$\Phi_t(\eta) = \varphi\Big(t, \eta(t), g_1(t,\eta), \ldots, g_m(t,\eta)\Big),$$

where φ is smooth and $g_1, \ldots, g_m$ are HINDSIGHT FACTORS.

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In this talk we extend the no-arbitrage result of [BSV] to strategies that include certain kind of stopping times. The key concept is local continuity.

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The allowed strategies are natural from the hedging point of view: Hedging strategies of typical options are of this type. However, from the no-arbitrage point of view the allowed strategies are not so natural: They do not include stopping times.

In this talk we extend the no-arbitrage result of [BSV] to strategies that include certain kind of stopping times. The key concept is LOCAL CONTINUITY.

While stopping times are rarely continuous, the author is not aware of any (reasonable) stopping time that is not locally continuous.

Outline

Locally Continuous Stopping Times

Definition (Local Continuity)

Let $\mathcal{X}$ and $\mathcal{Y}$ be metric spaces. A function $f : \mathcal{X} \to \mathcal{Y}$ is LOCALLY CONTINUOUS if for all $x \in \mathcal{X}$ there exists an open $U_x \subset \mathcal{X}$ such that $x \in \bar{U}_x$ and $f(x_n) \to f(x)$ whenever $x_n \to x$ in U_x.

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Example

An indicator $\mathbf{1}_A : \mathbb{R} \to \mathbb{R}$

1. is locally continuous if $A = \bar{G}$, G is open,
2. is not locally continuous if A has an isolated point.

Locally Continuous Stopping Times

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1. $\tau(\eta) = \inf\{t; \eta(t) \in F\}$, F is closed,
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In the case (3) above we say that τ is FAT. All the stopping times in the example above are fat.

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The functionals in the example above are locally continuous even if they were not stopping times.

Outline

No-Arbitrage with Stopping-Allowed Strategies

Definition (Stopping-Allowed Strategies)

A trading strategy Φ is Stopping-Allowed if it is of the form

$$\Phi_t = \sum_{k=1}^{n} \Phi_t^{(k)} \mathbf{1}_{(\tau_k, \tau_{k+1}]}(t),$$

where the $\Phi^{(k)}$'s are allowed and τ_k's are locally continuous.

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where the $\Phi^{(k)}$'s are allowed and τ_k's are locally continuous.

The definition above is understood in the conditional sense, i.e. $\Phi^{(k)}$ may depend on on $\mathcal{F}_{\tau_k}$ and $\tau_{k+1} \geq \tau_k$ is locally continuous in the conditioned, or quotient, space $\mathcal{C}_{S_{\tau_k},\sigma}[\tau_k, T]$.

No-Arbitrage with Stopping-Allowed Strategies

Theorem (No-Arbitrage with Stopping-Allowed Strategies)

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Lemma (No-Arbitrage with Take-the-Money-and-Run Strategies)

Let Φ be allowed strategy and let τ be a locally continuous stopping time. Then $\Phi \mathbf{1}_{[0,\tau]}$ is not an arbitrage opportunity.

No-Arbitrage with Stopping-Allowed Strategies

Proof of Lemma (No-Arbitrage with Take-the-Money-and-Run Strategies).

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Let $\Phi \mathbf{1}_{[0,\tau]}$ be a candidate for an arbitrage opportunity: $V_0(\Phi \mathbf{1}_{[0,\tau]}) = 0$ and $V_T(\Phi \mathbf{1}_{[0,\tau]}) \geq 0$ **P**-a.s., or

$$v(\tau(\eta), \eta; \varphi) \geq 0 \quad \text{for } \mathbf{P}\text{-a.a. } \eta.$$

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No-Arbitrage with Stopping-Allowed Strategies

Proof of Lemma (No-Arbitrage with Take-the-Money-and-Run Strategies).

Since $v(\tau(\eta), \eta; \varphi) \geq 0$ for all η we have in particular that $V_T(\Phi \mathbf{1}_{[0,\tau]}) \geq 0$ $\tilde{\mathbf{P}}$-a.s. ($\tilde{\mathbf{P}}$ stands for the Black-Scholes reference model).

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No-Arbitrage with Stopping-Allowed Strategies

Proof of Theorem (No-Arbitrage with Stopping-Allowed Strategies).

By using the conditional small-ball property instead of an unconditional one we see that Lemma (No-Arbitrage with Take-the-Money-and-Run Strategies) can be strengthened to:

$$\Phi^{(k)}\mathbf{1}_{(\tau_k,\tau_{k+1}]}$$

is not an arbitrage opportunity. Here the allowed strategy $\Phi^{(k)}$ may depend additionally on $\mathcal{F}_{\tau_k}$, and τ_{k+1} is locally continuous on the quotient, or conditioned, space $\mathcal{C}_{S_{\tau_k},\sigma}[\tau_k, T]$.

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But this means that the stopping-allowed strategy Φ does not generate arbitrage on any of the stochastic intervals $(\tau_k, \tau_{k+1}]$. Hence, it cannot generate arbitrage on the interval $[0, T]$. □

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If we assume that the trading strategy is constant between the switching stopping times we can weaken the local continuity assumption.

One way to weaken the assumption is to ask only local lower semi-continuity instead of local "full" continuity.

No-Arbitrage with Simple Strategies

Definition (Local Lower Semi-Continuity)

Let $\mathcal{X}$ be a metric space and let $\mathcal{Y}$ be an ordered complete metric space. A function $f : \mathcal{X} \to \mathcal{Y}$ is locally lower semi-continuous if for all $x \in \mathcal{X}$ there exists an open $U_x \subset \mathcal{X}$ such that $x \in \bar{U}_x$ and $\liminf f(x_n) \geq f(x)$ whenever $x_n \to x$ in U_x.

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Example

An indicator $\mathbf{1}_A : \mathcal{X} \to \mathbb{R}$ is locally lower semi-continuous if for all $x \in A$ and $\varepsilon > 0$ there exists a ball $B \subset A$ such that $\mathrm{dist}(x, B) < \varepsilon$.

No-Arbitrage with Simple Strategies

Definition (Simple Strategy)

A trading Φ strategy is SIMPLE if it is of the form

$$\Phi_t = \sum_{k=1}^{n} \xi_k \mathbf{1}_{(\tau_k, \tau_{k+1}]},$$

where τ_k is locally lower semi-continuous stopping times (relative to τ_{k-1}) and ξ_k's are $\mathcal{F}_{\tau_k}$ measurable.

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Theorem (No-Arbitrage with Simple Strategies)

Let Φ be a simple strategy. Then Φ is not an arbitrage opportunity.

No-Arbitrage with Simple Strategies

Before going to the proof of Theorem (No-Arbitrage with Simple Strategies) let us note that the theorem is true even without the assumption on the quadratic variation. In contrast, Theorem (No-Arbitrage with Stopping-Allowed Strategies) fails to be true if the quadratic variation vanishes.

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Before going to the proof of Theorem (No-Arbitrage with Simple Strategies) let us note that the theorem is true even without the assumption on the quadratic variation. In contrast, Theorem (No-Arbitrage with Stopping-Allowed Strategies) fails to be true if the quadratic variation vanishes.

Because of the “time-linearity” of the arbitrage and conditional small-ball property it is enough to show the following:

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Because of the "time-linearity" of the arbitrage and conditional small-ball property it is enough to show the following:

Lemma (Up'n'Down)

Let τ be locally lower semi-continuous stopping time. Then

$$\mathbf{P}[S_\tau > s_0] > 0 \quad \textit{and} \quad \mathbf{P}[S_\tau < s_0] > 0.$$

No-Arbitrage with Simple Strategies

Proof of Lemma (Up'n'Down).

We show that $\mathbf{P}[S_\tau > s_0] > 0$; the case $\mathbf{P}[S_\tau < s_0] > 0$ is symmetric.

No-Arbitrage with Simple Strategies

Proof of Lemma (Up'n'Down).

We show that $\mathbf{P}[S_\tau > s_0] > 0$; the case $\mathbf{P}[S_\tau < s_0] > 0$ is symmetric.

We show that the set $\{S_\tau > s_0\} = \{\eta; \eta(\tau(\eta)) > s_0\}$ contains a ball. Then the claim will follow from the small-ball property.

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Fix an increasing and concave path η_0 with $\eta_0(0) = s_0$ and a local lower semi-continuity set U_{η_0} of τ at η_0.

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Fix an increasing and concave path η_0 with $\eta_0(0) = s_0$ and a local lower semi-continuity set U_{η_0} of τ at η_0.

Since τ is lower semi-continuous on U_{η_0} we can find such an $\varepsilon < 1/2\,(\eta_0(\tau(\eta_0)) - s_0)$ that $\tau(\eta) \geq 1/2\,\tau(\eta_0)$ whenever $\eta \in B$, where B is some ball contained in $B_{\eta_0}(\varepsilon) \cap U_{\eta_0}$.

No-Arbitrage with Simple Strategies

Proof of Lemma (Up'n'Down).

Since η_0 is increasing and concave

$$\begin{aligned}
\eta(\tau(\eta)) &> \eta_0(\tau(\eta)) - 1/2\,(\eta_0(\tau(\eta_0)) - s_0) \\
&\geq \eta_0\,(1/2\,\tau(\eta_0)) - 1/2\,\eta_0(\tau(\eta_0)) + 1/2\,s_0 \\
&\geq 1/2\,\eta_0(0) + 1/2\,s_0 = s_0.
\end{aligned}$$

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$$\begin{aligned}
\eta(\tau(\eta)) &> \eta_0(\tau(\eta)) - 1/2\,(\eta_0(\tau(\eta_0)) - s_0) \\
&\geq \eta_0\,(1/2\,\tau(\eta_0)) - 1/2\,\eta_0(\tau(\eta_0)) + 1/2\,s_0 \\
&\geq 1/2\,\eta_0(0) + 1/2\,s_0 = s_0.
\end{aligned}$$

So, the ball B is contained in the set $\{S_\tau > s_0\}$, which implies that $\mathbf{P}[S_\tau > s_0] > 0$. $\square$

No-Arbitrage with Simple Strategies

Remark (ε-delay)

The Lemma (Up'n'Down) is true with local lower semi-continuity replaced by a weaker assumption of ε-DELAY:

For all η_0 there are positive $\varepsilon = \varepsilon(\eta_0)$ and $\delta = \delta(\eta_0)$ such that

$$\tau(\eta) \geq \varepsilon \quad \text{when} \;\; \eta \in B_{\eta_0}(\delta).$$

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- The End -