Local continuity

(for stopping times)

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1. Local Continuity

2. Stopping Times

3. Options, Arbitrage, and Replication

4. Market Models with Quadratic Variation and Small-Balls
Outline

1 Local Continuity

2 Stopping Times

3 Options, Arbitrage, and Replication

4 Market Models with Quadratic Variation and Small-Balls
**Local Continuity**

**Topological Definition**

**Definition (Local Continuity (Topological) ?)**

A function \( f : \mathcal{X} \to \mathcal{Y} \) between topological spaces is **locally continuous** at \( x \in \mathcal{X} \) if there exists a set \( U_x \subset \mathcal{X} \) such that

1. \( U_x \) is open,
2. \( x \in \overline{U}_x \),
3. for every neighbourhood \( V \) of \( f(x) \) there exists a neighbourhood \( W \) of \( x \) such that \( f(W \cap U_x) \subset V \).

**Remark**

The set \( W \cap U_x \) is a non-empty open set.
Local Continuity
Topological Definition

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(II) $x \in \overline{U}_x$,

(III) for every neighbourhood $V_{f(x)}$ of $f(x) \in \mathcal{Y}$ there exists a neighbourhood $W_x$ of $x \in \mathcal{X}$ such that

$$f [W_x \cap U_x] \subset V_{f(x)}.$$
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**Remark**

The set $W_x \cap U_x$ is a non-empty open set.
**Lemma (Key Lemma)**

Let $f : X \rightarrow \mathbb{R}$ be locally continuous at $x \in X$. Suppose that $f(x) > \alpha$. Then there is an open set $V \subset X$ such that $f(x') > \alpha$ for all $x' \in V$. 

Proof. The claim follows simply by noticing that $(\alpha, \infty)$ is a neighbourhood of $f(x)$. $\Box$
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**Proof.**

The claim follows simply by noticing that $(\alpha, \infty)$ is a neighbourhood of $f(x)$.

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Local Continuity
Metric Definition

**Definition (Local Continuity (Metric))**

Let $\mathcal{X}$ and $\mathcal{Y}$ be metric spaces. A function $f : \mathcal{X} \to \mathcal{Y}$ is **locally continuous** if for all $x \in \mathcal{X}$ there exists an open $U_x \subset \mathcal{X}$ such that $x \in \overline{U}_x$ and $f(x_n) \to f(x)$ whenever $x_n \to x$ in $U_x$. 
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**Remark (Local, Directional, and Proper Continuity)**

Local continuity at $x$ is continuity from the direction $U_x$. If $x \in U_x$ then local continuity is continuity.
**Local Continuity**  
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**Remark (Generalization to (Topological) Measure Spaces)**

One might want to consider local continuity in measure spaces. Then the **open** local continuity set is replaced by a **non-null** local continuity set.
Example (Simple One)

An indicator $1_A : \mathbb{R} \rightarrow \mathbb{R}$

1. is locally continuous if $A = \tilde{G}$, $G$ is open,
2. is not locally continuous if $A$ has an isolated point.
Local Continuity

Examples

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Example (Interesting One)

A functional $\tau : C[0, T] \to [0, T]$ defined by

$$\tau(\omega) = \min \{ t; \omega(t) = c \}$$

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**Example (Interesting One)**

A functional \(\tau : C[0, T] \to [0, T]\) defined by

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\tau(\omega) = \min \{t; \omega(t) = c\}
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is locally continuous. Indeed, for \(\omega_0 \in C[0, T]\), take

\[
U_{\omega_0} = \{\omega; \omega(t) > \omega_0(t) \text{ for all } t \in [0, T]\}.
\]
**Example**

Consider functions $f : \mathbb{R}^2 \to \mathbb{R}$. 
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1. \( f(x, y) = 1_{\{0\} \times [0, \infty)}(x, y) \)

   is directionally continuous at \((0, 0)\) along path \( \{(0, y); y \geq 0\} \), but not locally continuous at \((0, 0)\).
**Local Continuity**

**Local Continuity vs. Directional Continuity**

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2. \[
f(x, y) = \sum_{n=1}^{\infty} 1_{\left\{ 4^{-n-1} \leq \sqrt{x^2+y^2} \leq 4^{-n} \right\}}
\]
   is locally continuous at \((0, 0)\) but not directionally continuous along any path ending at \((0, 0)\).
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**Definition (Stopping Time)**

Let \( (\mathcal{F}_t)_{t \in [0, T]} \) be a flow of information. A random variable \( \tau : \Omega \to [0, T] \) is an \( (\mathcal{F}_t) \)-**STOPPING TIME** if \( \{\tau \leq t\} \in \mathcal{F}_t \) for all \( t \in [0, T] \).
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Let \((\mathcal{F}_t)\) be the information generated by observing a stochastic process \((S_t)\). Then

1. \(\tau(\omega) = \inf\{t; S_t(\omega) \geq c\}\) is a stopping time,
2. \(\tau(\omega) = \inf\{t; S_t(\omega) = \max_{u \in [0, T]} S_u(\omega)\}\) is not a stopping time.
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The functionals in the example above are locally continuous even if they were not stopping times.
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Let $S = (S_t)_{t \in [0,T]}$ be an asset-price process. We consider the canonical probability space, where $\Omega = C_+[0, T]$, $\mathcal{F}$ is its Borel-$\sigma$-algebra, and $\mathbf{P}$ is the distribution of $S$. So we have $S_t(\omega) = \omega(t)$.
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**Definition (Option)**

Option is simply a mapping $G : C_+[0, T] \rightarrow \mathbb{R}$. The asset $S$ is the **underlying** of the option $G$. 
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**Example**

- $G = (S_T - K)^+$ is a **call-option**, 
- $G = (K - S_T)^+$ is a **put-option**, 
- $G = S_T - K$ is a **future**.
A **trading strategy** $\Phi = (\Phi_t)_{t \in [0, T]}$ is an $S$-adapted stochastic process that tells the units of the underlying asset $S$ the investor has is her portfolio at any time $t \in [0, T]$. 

**Definition (Arbitrage)**

Arbitrage is a trading strategy $\Phi$ with the properties:

- $V_0(\Phi) = 0$,
- $V_t(\Phi) \geq 0$ for all $t \in [0, T]$, and
- $P[V_T(\Phi) > 0] > 0$.

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The **wealth** of the trading strategy $\Phi$ is (in the discounted world) satisfies

$$dV_t(\Phi) = \Phi_t \, dS_t,$$

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**Definition (Replication principle)**

Let $G$ be an option. Suppose that there is a trading strategy $\Phi$ with initial wealth $V_0(\Phi)$ such that $G = V_T(\Phi)$. Then the price of the option $G$ is $V_0(\Phi)$. 

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The replication requirement $G = V_T(\Phi)$ can be written as

$$G = V_0(\Phi) + \int_0^T \Phi_t \, dS_t,$$

where the integral is of “forward type”.
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Assume that $S$ is continuous, strictly positive, starts from $s_0$, and the information used in trading is generated by it.
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Assume that $S$ has the **QUADRATIC VARIATION**

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Assume the **CONDITIONAL SMALL-BALL PROPERTY**

$$\mathbb{P} \left[ \sup_{t \in [\tau, T]} |S_t - \omega(t)| < \varepsilon \left| \mathcal{F}_\tau \right. \right] > 0$$

$\mathbb{P}$-a.s. for all paths $\omega$, positive $\varepsilon$, and stopping times $\tau$. 
Market Models with Quadratic Variation and Small-Balls Canonical Space

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$$P \left[ \sup_{t \in [\tau, T]} |S_t - \omega(t)| < \varepsilon \mid \mathcal{F}_\tau \right] > 0$$

$P$-a.s. for all paths $\omega$, positive $\varepsilon$, and stopping times $\tau$.

So, we have a collection of models $P$ on the canonical filtered space $C_{s_0,\sigma}[0, T]$, where $P$ is restricted only by the assumptions of quadratic variation and conditional small-ball property.
[BSV]$^1$ showed that with \textit{ALLOWED} strategies that depend smoothly on time, spot, running maximum, running minimum and such one cannot do arbitrage.

$^1$Bender, S., Valkeila: \textit{No-arbitrage pricing beyond semimartingales}. WIAS Preprint No. 1110, 2006.
[BSV]$^1$ showed that with **ALLOWED** strategies that depend smoothly on time, spot, running maximum, running minimum and such one cannot do arbitrage.

The result followed from the fact that

$$V_t(\Phi)(\omega) = V_0(\Phi)(\omega) + v(t, \omega; \varphi) \quad \text{for } P\text{-a.a. } \omega,$$

where $v(t, \omega; \varphi)$ is continuous in $\omega$ uniformly in $t$. Here $\varphi$ is the strategy functional associated to $\Phi$:

$$\Phi_t(\omega) = \varphi(t, \omega(t), g_1(t, \omega), \ldots, g_m(t, \omega)),$$

where $\varphi$ is smooth and $g_1, \ldots, g_m$ are **HINDSIGHT FACTORS**.

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The allowed strategies are natural from the hedging point of view: Hedging strategies of typical options are of this type. However, from the no-arbitrage point of view the allowed strategies are not so natural: They do not include stopping times.
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We can extend the no-arbitrage result of [BSV] to strategies that include **locally continuous** stopping times.

While stopping times are rarely continuous, the author is not aware of any (reasonable) stopping times that are not locally continuous.
Definition (Stopping-Allowed Strategies)

A trading strategy $\Phi$ is **Stopping-Allowed** if it is of the form

$$\Phi_t = \sum_{k=1}^{n} \Phi^{(k)}_t 1_{(\tau_k, \tau_{k+1}]}(t),$$

where the $\Phi^{(k)}$'s are allowed and $\tau_k$'s are locally continuous.
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where the $\Phi^{(k)}$’s are allowed and $\tau_k$’s are locally continuous.

The definition above is understood in the conditional sense, i.e. $\Phi^{(k)}$ may depend on $\mathcal{F}_{\tau_k}$ and $\tau_{k+1} \geq \tau_k$ is locally continuous in the conditioned, or quotient, space $C_{\tau_k', \sigma} [\tau_k, T]$. 
Theorem (No-Arbitrage with Stopping-Allowed Strategies)

Let $\Phi$ be a stopping-allowed strategy. Then $\Phi$ is not an arbitrage opportunity.
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Theorem (No-Arbitrage with Stopping-Allowed Strategies) follows by applying the conditional small-ball property $n$ times with the following lemma:
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Theorem (No-Arbitrage with Stopping-Allowed Strategies) follows by applying the conditional small-ball property \( n \) times with the following lemma:

Lemma (No-Arbitrage with Take-the-Money-and-Run Strategies)

Let \( \Phi \) be allowed strategy and let \( \tau \) be a locally continuous stopping time. Then \( \Phi 1_{[0,\tau]} \) is not an arbitrage opportunity.

Let $\Phi_{[0, \tau]}$ be a candidate for an arbitrage opportunity:

$V_0(\Phi_{[0, \tau]}) = 0$ and $V_T(\Phi_{[0, \tau]}) \geq 0$ $\mathbb{P}$-a.s., or

$v(\tau(\omega), \omega; \phi) \geq 0$ for $\mathbb{P}$-a.a. $\omega$.

We show that $v(\tau(\omega), \omega; \phi) \geq 0$ for all $\omega$:

Suppose that $v(\tau(\omega_0), \omega_0; \phi) < 0$ for some $\omega_0$.

Let $U_{\omega_0}$ be the local continuity set of $\tau$ at $\omega_0$. Since $v(t, \cdot; \phi)$ is continuous uniformly in $t$ we see that $v(\tau(\cdot), \cdot; \phi)$ is continuous on $U_{\omega_0}$.

So, there must be a ball $B \subset U_{\omega_0}$ such that $v(\tau(\omega), \omega; \phi) < 0$ for all $\omega \in B$.

But due to the small-ball property this means that $\mathbb{P}[V_T(\Phi_{[0, \tau]}) < 0] > 0$, which is a contradiction.

Let $\Phi_{1[0,\tau]}$ be a candidate for an arbitrage opportunity: $V_0(\Phi_{1[0,\tau]}) = 0$ and $V_T(\Phi_{1[0,\tau]}) \geq 0$ $\mathbb{P}$-a.s., or

$$v(\tau(\omega), \omega; \varphi) \geq 0 \quad \text{for } \mathbb{P}\text{-a.a. } \omega.$$

Let $\Phi_1^{[0,\tau]}$ be a candidate for an arbitrage opportunity: $V_0(\Phi_1^{[0,\tau]}) = 0$ and $V_\tau(\Phi_1^{[0,\tau]}) \geq 0$ $\mathbb{P}$-a.s., or

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\]

We show that \( \nu(\tau(\omega), \omega; \varphi) \geq 0 \) for all \( \omega \): Suppose that 
\( \nu(\tau(\omega_0), \omega_0; \varphi) < 0 \) for some \( \omega_0 \).

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We show that $v(\tau(\omega), \omega; \varphi) \geq 0$ for all $\omega$: Suppose that $v(\tau(\omega_0), \omega_0; \varphi) < 0$ for some $\omega_0$. Let $U_{\omega_0}$ be the local continuity set of $\tau$ at $\omega_0$. Since $v(t, \cdot; \varphi)$ is continuous uniformly in $t$ we see that $v(\tau(\cdot), \cdot; \varphi)$ is continuous on $U_{\omega_0}$. 


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Proof of Lemma (No-Arbitrage with Take-the-Money-and-Run Strategies), contd.

Since $v(\tau(\omega), \omega; \varphi) \geq 0$ for all $\omega$ we have in particular that $V_T(\Phi 1_{[0,\tau]} \geq 0 \tilde{\mathbb{P}}$-a.s. ($\tilde{\mathbb{P}}$ stands for the Black-Scholes reference model).
Proof of Lemma (No-Arbitrage with Take-the-Money-and-Run Strategies), contd.

Since \( \nu(\tau(\omega), \omega; \varphi) \geq 0 \) for all \( \omega \) we have in particular that \( V_T(\Phi 1_{[0, \tau]} \geq 0 \) \( \tilde{\mathcal{P}} \)-a.s. (\( \tilde{\mathcal{P}} \) stands for the Black-Scholes reference model). The classical theory then tells us that \( V_T(\Phi 1_{[0, \tau]} = 0 \) \( \tilde{\mathcal{P}} \)-a.s.
Proof of Lemma (No-Arbitrage with Take-the-Money-and-Run Strategies), contd.

Since $\nu(\tau(\omega), \omega; \varphi) \geq 0$ for all $\omega$ we have in particular that $V_T(\Phi 1_{[0, \tau]}) \geq 0$ $\tilde{P}$-a.s. ($\tilde{P}$ stands for the Black-Scholes reference model). The classical theory then tells us that $V_T(\Phi 1_{[0, \tau]}) = 0$ $\tilde{P}$-a.s. Then, by using the local continuity as before, we see that $\nu(\tau(\omega), \omega; \varphi) = 0$ for all $\omega$. 
Since \( \nu(\tau(\omega), \omega; \varphi) \geq 0 \) for all \( \omega \) we have in particular that \( V_T(\Phi 1_{[0,\tau]}) \geq 0 \tilde{\mathbb{P}}\)-a.s. (\( \tilde{\mathbb{P}} \) stands for the Black-Scholes reference model). The classical theory then tells us that \( V_T(\Phi 1_{[0,\tau]}) = 0 \tilde{\mathbb{P}}\)-a.s. Then, by using the local continuity as before, we see that \( \nu(\tau(\omega), \omega; \varphi) = 0 \) for all \( \omega \). But this means that \( V(\Phi 1_{[0,\tau]}) = 0 \mathbb{P}\)-a.s. So, \( \Phi 1_{[0,\tau]} \) is not an arbitrage opportunity.
Proof of Theorem (No-Arbitrage with Stopping-Allowed Strategies).

By using the conditional small-ball property instead of an unconditional one we see that Lemma (No-Arbitrage with Take-the-Money-and-Run Strategies) can be strengthened to:

\[ \Phi^{(k)}1_{(\tau_k, \tau_{k+1}]} \]

is not an arbitrage opportunity. Here the allowed strategy \( \Phi^{(k)} \) may depend additionally on \( \mathcal{F}_{\tau_k} \), and \( \tau_{k+1} \) is locally continuous on the quotient, or conditioned, space \( \mathcal{C}_{S_{\tau_k, \sigma}[\tau_k, T]} \).
Proof of Theorem (No-Arbitrage with Stopping-Allowed Strategies).

By using the conditional small-ball property instead of an unconditional one we see that Lemma (No-Arbitrage with Take-the-Money-and-Run Strategies) can be strengthened to:

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But this means that the stopping-allowed strategy \( \Phi \) does not generate arbitrage on any of the stochastic intervals \( (\tau_k, \tau_{k+1}] \). Hence, it cannot generate arbitrage on the interval \([0, T]\). \( \square \)
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