Local continuity

(for stopping times)

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Outline

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DEFINITION (LOCAL CONTINUITY (TOPOLOGICAL) ?)

A function $f : \mathcal{X} \to \mathcal{Y}$ between topological spaces is LOCALLY CONTINUOUS AT $x \in \mathcal{X}$ if there exists a set $U_x \subset \mathcal{X}$ such that

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(II) $x \in \bar{U}_x$,

(III) for every neighbourhood $V_{f(x)}$ of $f(x) \in \mathcal{Y}$ there exists a neighbourhood W_x of $x \in \mathcal{X}$ such that

$$f\left[W_x \cap U_x\right] \subset V_{f(x)}.$$

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$$f\left[W_x \cap U_x\right] \subset V_{f(x)}.$$

REMARK

The set $W_x \cap U_x$ is a non-empty open set.

Lemma (Key Lemma)

Let $f : \mathcal{X} \to \mathbb{R}$ be locally continuous at $x \in \mathcal{X}$. Suppose that $f(x) > \alpha$. Then there is an open set $V \subset X$ such that $f(x') > \alpha$ for all $x' \in V$.

Lemma (Key Lemma)

Let $f : \mathcal{X} \to \mathbb{R}$ be locally continuous at $x \in \mathcal{X}$. Suppose that $f(x) > \alpha$. Then there is an open set $V \subset X$ such that $f(x') > \alpha$ for all $x' \in V$.

Proof.

The claim follows simply by noticing that (α, ∞) is a neighbourhood of $f(x)$. □

DEFINITION (LOCAL CONTINUITY (METRIC))

Let $\mathcal{X}$ and $\mathcal{Y}$ be metric spaces. A function $f : \mathcal{X} \to \mathcal{Y}$ is LOCALLY CONTINUOUS if for all $x \in \mathcal{X}$ there exists an open $U_x \subset \mathcal{X}$ such that $x \in \bar{U}_x$ and $f(x_n) \to f(x)$ whenever $x_n \to x$ in U_x.

Local Continuity
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Remark (Local, Directional, and Proper Continuity)

Local continuity at x is continuity from the direction U_x. If $x \in U_x$ then local continuity is continuity.

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Local continuity at x is continuity from the direction U_x. If $x \in U_x$ then local continuity is continuity.

Remark (Generalization to (Topological) Measure Spaces)

One might want to consider local continuity in measure spaces. Then the OPEN local continuity set is replaced by a NON-NULL

Example (Simple One)

An indicator $\mathbf{1}_A : \mathbb{R} \to \mathbb{R}$

1. is locally continuous if $A = \bar{G}$, G is open,
2. is not locally continuous if A has an isolated point.

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is locally continuous. Indeed, for $\omega_0 \in C[0, T]$, take

$$U_{\omega_0} = \{\omega; \omega(t) > \omega_0(t) \text{ for all } t \in [0, T]\}.$$

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2.
$$f(x,y) = \sum_{n=1}^{\infty} \mathbf{1}_{\left\{4^{-n-1} \leq \sqrt{x^2+y^2} \leq 4^{-n}\right\}}$$
is locally continuous at $(0,0)$ but not directionally continuous along any path ending at $(0,0)$.

Outline

Definition (Stopping Time)

Let $(\mathcal{F}_t)_{t\in[0,T]}$ be a flow of information. A random variable $\tau : \Omega \to [0, T]$ is an $(\mathcal{F}_t)$-stopping time if $\{\tau \le t\} \in \mathcal{F}_t$ for all $t \in [0, T]$.

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Example

Let $(\mathcal{F}_t)$ be the information generated by observing a stochastic process (S_t). Then

1. $\tau(\omega) = \inf\{t; S_t(\omega) \geq c\}$ is a stopping time,
2. $\tau(\omega) = \inf\{t; S_t(\omega) = \max_{u\in[0,T]} S_u(\omega)\}$ is not a stopping time.

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The functionals in the example above are locally continuous even if they were not stopping times.

Outline

Let $S = (S_t)_{t \in [0,T]}$ be an asset-price process. We consider the canonical probability space, where $\Omega = C_+[0, T]$, $\mathcal{F}$ is its Borel-σ-algebra, and $\mathbf{P}$ is the distribution of S. So we have $S_t(\omega) = \omega(t)$.

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Option is simply a mapping $G : C_+[0, T] \to \mathbb{R}$. The asset S is the underlying of the option G.

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Example

- $G = (S_T - K)^+$ is a call-option,
- $G = (K - S_T)^+$ is a put-option,
- $G = S_T - K$ is a future.

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The wealth of the trading strategy Φ is (in the discounted world) satisfies

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Definition (Arbitrage)

Arbitrage is a trading strategy Φ with the properties: $V_0(\Phi) = 0$, $V_t(\Phi) \geq 0$ for all $t \in [0, T]$, and $\mathbf{P}[V_T(\Phi) > 0] > 0$.

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Let G be an option. Suppose that there is a trading strategy Φ with initial wealth $V_0(\Phi)$ such that $G = V_T(\Phi)$. Then the price of the option G is $V_0(\Phi)$.

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The replication requirement $G = V_T(\Phi)$ can be written as

$$G \ = \ V_0(\Phi) + \int_0^T \Phi_t \, \mathrm{d}S_t,$$

where the integral is of "forward type".

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Assume the CONDITIONAL SMALL-BALL PROPERTY

$$\mathbf{P}\left[\sup_{t\in[\tau,T]} |S_t - \omega(t)| < \varepsilon \,\middle|\, \mathcal{F}_\tau\right] > 0$$

$\mathbf{P}$-a.s. for all paths ω, positive ε, and stopping times τ.

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So, we have a collection of models **P** on the canonical filtered space $\mathcal{C}_{s_0,\sigma}[0, T]$, where **P** is restricted only by the assumptions of quadratic variation and conditional small-ball property.

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[BSV][1] showed that with ALLOWED strategies that depend smoothly on time, spot, running maximum, running minimum and such one cannot do arbitrage.

The result followed from the fact that

$$V_t(\Phi)(\omega) = V_0(\Phi)(\omega) + v(t,\omega;\varphi) \quad \text{for } \mathbf{P}\text{-a.a. } \omega,$$

where $v(t,\omega;\varphi)$ is continuous in ω uniformly in t. Here φ is the strategy functional associated to Φ:

$$\Phi_t(\omega) = \varphi\Big(t, \omega(t), g_1(t,\omega), \ldots, g_m(t,\omega)\Big),$$

where φ is smooth and $g_1, \ldots, g_m$ are HINDSIGHT FACTORS.

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We can extend the no-arbitrage result of [BSV] to strategies that include LOCALLY CONTINUOUS stopping times.

While stopping times are rarely continuous, the author is not aware of any (reasonable) stopping times that are not locally continuous.

Definition (Stopping-Allowed Strategies)

A trading strategy Φ is Stopping-Allowed if it is of the form

$$\Phi_t = \sum_{k=1}^{n} \Phi_t^{(k)} \mathbf{1}_{(\tau_k, \tau_{k+1}]}(t),$$

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where the $\Phi^{(k)}$'s are allowed and τ_k's are locally continuous.

The definition above is understood in the conditional sense, i.e. $\Phi^{(k)}$ may depend on on $\mathcal{F}_{\tau_k}$ and $\tau_{k+1} \geq \tau_k$ is locally continuous in the conditioned, or quotient, space $\mathcal{C}_{S_{\tau_k},\sigma}[\tau_k, T]$.

Theorem (No-Arbitrage with Stopping-Allowed Strategies)

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Theorem (No-Arbitrage with Stopping-Allowed Strategies) follows by applying the conditional small-ball property n times with the following lemma:

Lemma (No-Arbitrage with Take-the-Money-and-Run Strategies)

Let Φ be allowed strategy and let τ be a locally continuous stopping time. Then $\Phi \mathbf{1}_{[0,\tau]}$ is not an arbitrage opportunity.

Proof of Lemma (No-Arbitrage with Take-the-Money-and-Run Strategies).

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PROOF OF LEMMA (NO-ARBITRAGE WITH TAKE-THE-MONEY-AND-RUN STRATEGIES).

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Proof of Lemma (No-Arbitrage with Take-the-Money-and-Run Strategies), contd.

Since $v(\tau(\omega), \omega; \varphi) \geq 0$ for all ω we have in particular that $V_T(\Phi \mathbf{1}_{[0,\tau]}) \geq 0$ $\tilde{\mathbf{P}}$-a.s. ($\tilde{\mathbf{P}}$ stands for the Black-Scholes reference model).

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Proof of Theorem (No-Arbitrage with Stopping-Allowed Strategies).

By using the conditional small-ball property instead of an unconditional one we see that Lemma (No-Arbitrage with Take-the-Money-and-Run Strategies) can be strengthened to:

$$\Phi^{(k)}\mathbf{1}_{(\tau_k,\tau_{k+1}]}$$

is not an arbitrage opportunity. Here the allowed strategy $\Phi^{(k)}$ may depend additionally on $\mathcal{F}_{\tau_k}$, and τ_{k+1} is locally continuous on the quotient, or conditioned, space $\mathcal{C}_{S_{\tau_k},\sigma}[\tau_k, T]$.

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But this means that the stopping-allowed strategy Φ does not generate arbitrage on any of the stochastic intervals $(\tau_k, \tau_{k+1}]$. Hence, it cannot generate arbitrage on the interval $[0, T]$. □

- The End -